# Risk contributions and performance measurement[*]

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## Abstract

Risk adjusted performance measurement for a portfolio involves calculating the risk contribution of each single asset. We show that there is only one definition for the risk contributions which is suitable for performance measurement, namely as derivative of the underlying risk measure in direction of the considered asset weight. We also compute the derivatives for some popular risk measures including the quantile-based value at risk (VaR) in a rather general context. As a consequence we obtain a mean-quantile CAPM.

*Keywords: Performance measurement, portfolio selection, value at risk (VaR), quantile, shortfall, capital asset pricing model (CAPM).*

# 1 Introduction

Suppose that an investor wants to place a fixed amount of capital into some asset. He has got two exclusive choices: asset $j, j = 1, 2$, yields the expected return $m_j$ with risk $r_j$. Evidently, if the risks are equal, he will choose the asset with the higher yield. In case of different risks (say $r_1 < r_2$) there is not such an obvious answer. No doubt that $m_2$ must be higher than $m_1$ for asset 2 to be eligible at all. But how large should be the difference in returns to make asset 2 more attractive?

The Markowitz portfolio theory (cf. [17]) is the classical reference for a solution to the problem. It is still effective nowadays (cf. [21]). Below, we will consider the theory in some detail. At the moment, we only note that Markowitz' notion of *risk* is rather abstract. It is considered a measure of uncertainty in return, and is defined mathematically as standard deviation of the return.

Despite its computational convenience this perspective has some drawbacks. From the technical point of view, it is not desirable that not only the unfavorable but also the

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favorable fluctuations of the return around its mean have an impact on its standard deviation. The way to handle this problem is clear: one has to make use of other definitions of risk. We will see some examples in Section 3.

But the economic interpretation of risk as standard deviation is difficult, as well. Is risk really an immaterial quantity? A tangible explanation of the notion could be useful. Let us consider a highly leveraged investor, i.e. an investor financing his assets largely with borrowed money. Above all, here the investment risk is creditors' risk: if the assets do not yield enough the investor has to fall back on his equity in order to meet his obligations; as soon as the equity is used up he goes bankrupt, and the creditors suffer losses.

This example suggests defining investment risk as the amount of equity that ensures with sufficient certainty the investor's capability to pay off his debt. With other words: risk is a capital reserve for preventing insolvency. In case of financial institutions – which are typically highly leveraged – the notion might be more restrictive. For instance, in [18] (p. 9), the "role of capital in a bank" is described as acting "as a buffer against future, unidentified, even relatively improbable losses, whilst still leaving the bank able to operate at the same level of capacity". A bank's creditors are typically depositors; in case of an insurance company they are policy holders, and in case of hedge funds they might be other financial institutions.

Equating risk to investor's required capital supplies an easy solution to the problem, formulated at the beginning, of finding the more profitable among two assets. Simply compute the return by $\frac{m_j}{r_j}$, $j = 1, 2$. Hence the investor should decide in favor of asset 1 if and only if $\frac{m_1}{r_1} \geq \frac{m_2}{r_2}$. This kind of computing the return is commonly called RORAC ("Return on Risk-Adjusted Capital", cf. [18] p. 59), and the comparison procedure is called "Risk-Adjusted Performance Measurement".

As a matter of course, there is now a new problem: how to determine the amount of capital that is necessary for preserving solvency? Here the notion of *value at risk* (VaR) offers its services: the amount of capital – often called *economic capital* – must be so large that the investor's solvency is ensured at level $\alpha$ (for instance $\alpha = 99$ %). VaR has become particularly popular since the Basle Committee on Banking Supervision ([4]) permitted banks to make use of it in their internal models for the capital required by market risks. As long as the returns (or log-returns) of a portfolio are normally distributed – implicitly, this is often assumed for market risk portfolios – the VaR method and the Markowitz theory yield identical results when applied for portfolio optimization. However, the normal distribution assumption cannot be uphold even for market risks as soon as the portfolio includes underlyings *and* derivative instruments. This observation is evident for instance from the simulation results in [20], Sec. 2. The normal distribution assumption seems to be completely wrong in case of a credit portfolio. For this and other reasons the Basle Committee for Banking Supervision hesitates to permit banks to make use of VaR models for calculating economic capital (cf. [5]). Nonetheless, some standard software packages for credit portfolio management enable its users to optimize portfolios only by variance-

based Markowitz methods; at the same time they compute the economic capital as VaR at a given confidence level (see [11] or [16]).

In this paper we examine how to combine capital allocation via VaR (or via other risk measures) and portfolio optimization in a compatible manner. The notion of risk contribution is crucial for doing so. Determining risk contributions means to apportion the economic capital of a portfolio onto the assets in a way preserving the differences in their riskiness. We show that there is only one definition for the risk contributions which is suitable for performance measurement, namely as derivative of the underlying risk measure in direction of the asset weight in question.

Evidently, this rises the question under which conditions risk measures are differentiable. We will discuss this problem for some popular risk measures: the standard deviation, the VaR and the shortfall. We will also review for these examples some of the risk contributions which have been proposed in the literature (cf. [1], [15] or [18]), and examine whether they are suitable for performance measurement.

The result on the differentiation of the VaR allows us to formulate a variant of the capital asset pricing model (CAPM) in which the variance-based $\beta$'s are replaced by quantile-based quantities.

This paper is organized as follows: after presenting our portfolio model in Section 2 we introduce our running examples of risk measures in Section 3. Section 4 contains the above-mentioned result on suitableness for performance measurement (Theorem 4.4). Section 5 gives the results on differentiation, and in Section 6 we discuss the connection to the Markowitz theory and the CAPM. We conclude in Section 7 with some summarizing remarks.

In the sequel we will make use of the following notation. For a positive integer $d$ the set $N_d$ is defined by $N_d \overset{\text{def}}{=} \{1, \ldots, d\}$. For a vector $x \in \mathbb{R}^d$, $x_i$ denotes its $i$-th component. For $x, y \in \mathbb{R}^d$ we denote by $x'y \overset{\text{def}}{=} \sum_{i=1}^d x_i \, y_i$ the Euclidean scalar product of $x$ and $y$. For $i \in N_d$ the vector $e^{(i)} \in \mathbb{R}^d$ denotes the $i$-th canonical unit vector, i.e.

$$e_j^{(i)} \quad = \quad \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \,. \end{cases} .$$

## 2   Background and model

We are going to study a model for the cash flow generated by an investment consisting of several assets $1, \ldots, d \geq 2$. We use the term asset as an abbreviation for "asset or liability" or the difference of these two. Thus, the cashflow of an asset may be positive and negative.

Examples for such assets are a risky loan granted by a bank and refinanced with deposits and a credit derivative on the default of the loan that the bank bought with borrowed money in order to reduce its risk. For the first asset the expected cash flow should be positive whereas for the second asset it might be negative.

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Mathematically we describe the cash flow $C_i$ of asset $i$ by its expected profit/loss margin $m_i$ and by $(-1)$ times the deviation of the cash flow from its margin. This means

$$(2.1) \qquad\qquad C_i \quad = \quad m_i - X_i$$

where $X_i$ is an integrable random variable with $\mathrm{E}\,[\,X_i\,] = 0$. We call $X_i$ the *fluctuation* caused by asset $i$. The cash flow from the investment now is

$$(2.2) \qquad\qquad C(u) \quad = \quad \sum_{i=1}^{d} u_i\, C_i \quad = \quad \sum_{i=1}^{d} m_i\, u_i - \sum_{i=1}^{d} u_i\, X_i$$

for an investment portfolio consisting of $u_i$ units of asset $i, i = 1, \ldots, d$. The random variable

$$(2.3) \qquad\qquad Z(u) \quad = \quad \sum_{i=1}^{d} u_i\, X_i$$

is the *portfolio (cash flow) fluctuation*.

In case of a negative cash flow $C(u)$ the investor will go bankrupt unless he has allocated some capital from his equity in order to prevent insolvency. The amount of capital allocated for this reason is called *economic capital*. This is the way equity contributes to the investment (cf. [18] p. 32). Hence the expected return on equity for the investment has to be calculated as a RORAC by the ratio of the expected cash flow and the economic capital. This shows that the economic capital is crucial for the performance of the investment. If it is low, the expected performance will be good but the probability of insolvency might also be high. If the economic capital is high the investor's creditors will be happy but the performance of the investment may be poor.

Thus the fact that there are a lot of suggestions for the definition of the economic capital is not astonishing. Each proposal has its advantages and disadvantages (cf. the discussions for risk measures under various aspects in [3], [2] or [20]).

We will distinguish the risk and the economic capital of a portfolio. The risk will be a quantity measuring the portfolio fluctuation as defined by (2.3) whereas the economic capital will depend on the portfolio fluctuation *and* on the profit/loss margins. In other words, while the risk will tell us only something about the deviations of the portfolio cash flow from its expected value, the economic capital will also take into account the expected value itself.

We do not need a formal definition of the notion *risk*. As seen above, a portfolio is represented by a vector $u = (u_1, \ldots, u_d) \in U \subset \mathbb{R}^d$. The $u_i$ may be interpreted as weights or numbers of pieces of the assets. The set $U$ contains the portfolios that are currently under consideration. A *risk measure* then is simply a function $r : U \to \mathbb{R}$, and $r(u)$ is the *risk* of portfolio $u$. We do not impose any special property on the function $r$ to be a risk measure, but we will often assume the risk measures to be differentiable functions. Also, when examining which risk measure to use for a portfolio in practice one might be well-advised to take care of some or of all the properties discussed in [3].

We need not define formally the economic capital of a portfolio either. Example 3.2 will show that the choice $r(u) - \sum_{i=1}^{d} u_i \, m_i$ for the economic capital is reasonable since with an appropriate risk measure $r$ the probability of the cash flow to fall short of $(-1)$ times this quantity will be low.

# 3   Examples of risk measures

Mapping the riskiness of a set $U$ of portfolios by a single function $r : U \to \mathbb{R}$ is not a simple task. Some knowledge about the portfolio is needed. Examples for the necessary knowledge are worst-case scenarios based on human expertise or statistical models of the portfolio cash flow which might be built from historical data. We will focus on the following three examples of risk measures from practice which all need a statistical model of the cash flow. Note that in these examples from a technical point of view the assumption $\mathrm{E}[X_i] = 0$ is not necessary. Nonetheless, it might be reasonable in an economical context (cf. Section 2).

**Example 3.1** (Standard deviation)
Assume that $(X_1, \ldots, X_d)$ is a random vector such that $\mathrm{var}(X_i) < \infty$ for each $i \in N_d$. Fix $c > 0$. With $Z(u)$ as in (2.3),

$$(3.1) \qquad r(u) \quad \overset{\text{def}}{=} \quad c \, \sqrt{\mathrm{var}(Z(u))}, \quad u \in \mathbb{R}^d,$$

defines the usual standard deviation risk measure which is very popular in practice. The constant $c$ is often chosen as the 95%- or the 99%-quantile of the standard normal distribution, i.e. $c = 1.65$ and $c = 2.33$ resp. □

**Example 3.2** (Value at risk)
Let $(X_1, \ldots, X_d)$ be any random vector in $\mathbb{R}^d$. Fix $\alpha \in (0, 1)$ and denote by

$$(3.2) \qquad Q_\alpha(u) \quad \overset{\text{def}}{=} \quad \inf\{z \in \mathbb{R} : \mathrm{P}[Z(u) \le z] \ge \alpha\}, \quad u \in \mathbb{R}^d,$$

the $\alpha$-*quantile* of the portfolio fluctuation $Z(u)$, defined by (2.3). Then

$$(3.3) \qquad r(u) \quad \overset{\text{def}}{=} \quad Q_\alpha(u), \quad u \in \mathbb{R}^d,$$

defines the risk measure "value at risk" (VaR). □

There does not seem to be any common view in the literature whether for the definition of VaR one should take the "pure" quantile or the quantile minus some benchmark. In Definition 4.1 we will use for arbitrary risk measures the expected profit/loss margin as a benchmark when defining our notion of return. The reason for doing so is the fact that

$$\mathrm{P}[C(u) < m'u - Q_\alpha(u)] \quad \le \quad 1 - \alpha,$$

with $C(u)$ being the portfolio cash flow as in (2.2). Hence $Q_\alpha(u) - m'u$ is just the amount of capital to be allocated in order to prevent insolvency with probability $\alpha$ or more.

**Example 3.3** (Shortfall)

Fix an integer $n \geq 1$ and an $\alpha \in (0, 1)$. Assume that $(X_1, \ldots, X_d)$ is a random vector such that $\mathrm{E}\,[\,|\,X_i\,|^n\,] < \infty$ for each $i \in N_d$.

Observe that for all $u \in \mathbb{R}^d$ we have by definition of $Q_\alpha(u)$ in (3.2)

$$(3.4) \qquad\qquad \mathrm{P}\,[\,Z(u) \geq Q_\alpha(u)\,] \quad \geq \quad 1 - \alpha\,.$$

Hence the risk measure

$$(3.5) \qquad\qquad r^{(n)}(u) \quad \overset{\mathrm{def}}{=} \quad \mathrm{E}\,[\,Z(u)^n \mid Z(u) \geq Q_\alpha(u)\,], \quad u \in \mathbb{R}^d,$$

is well-defined. It corresponds to the shortfall risk measure well-known from literature (cf. [2] for $n = 1$ or [20]). In Section 6 we will see that it might be reasonable to use $\sqrt[n]{r^{(n)}}$ instead of $r^{(n)}$. $\qquad\square$

In the Markowitz portfolio theory (cf. [21]) the fact that the standard deviation risk measure has nice differentiation properties is heavily exploited. In Section 5 we will see that the VaR and shortfall risk measures are differentiable in a rather general context as well.

From general mathematical analysis it is clear that the derivatives of a function play the most important role when we study the effects of changing the values of one or more of its arguments. For the class of homogeneous risk measures this connection is particularly close as the subsequent proposition shows. We need a slightly more general notion of homogeneity.

**Definition 3.4**

    *(i) A set $U \subset \mathbb{R}^d$ is* homogeneous *if for each $u \in U$ and $t > 0$ we have $t\,u \in U$.*

    *(ii) Let $\tau$ be any fixed real number. A function $r : U \to \mathbb{R}$ is $\tau$-*homogeneous *if $U$ is homogeneous and for each $u \in U$ and $t > 0$ we have $\quad t^\tau r(u) = r(t\,u)$.* $\qquad\square$

Proposition 3.5 tells us in (3.6) that differentiable $\tau$-homogeneous functions can be represented as a weighted sum of their derivatives in a canonical manner.

**Proposition 3.5**

*Let $\emptyset \neq U$ be a homogeneous open set in $\mathbb{R}^d$ and $r : U \to \mathbb{R}$ be a real-valued function. Let $\tau \in \mathbb{R}$ be fixed.*

a) *If $r$ is $\tau$-homogeneous and partially differentiable in $u_i$ for some $i \in N_d$ then the derivative $\frac{\partial r}{\partial u_i}$ is $(\tau - 1)$-homogeneous.*

b) *If $r$ is totally differentiable then it is $\tau$-homogeneous if and only if for all $u \in U$*

$$(3.6) \qquad \tau\, r(u) \quad = \quad \sum_{i=1}^{d} u_i\, \frac{\partial r}{\partial u_i}(u)\,.$$

c) *Assume $d \geq 2$. Let $r$ be $\tau$-homogeneous, continuous, and for $i = 2,\ldots,d$ partially differentiable in $u_i$ with continuous derivatives $\frac{\partial r}{\partial u_2},\ldots,\frac{\partial r}{\partial u_d}$. Then on the set $U \setminus \left( \{0\} \times \mathbb{R}^{d-1} \right)$ the function $r$ is also partially differentiable in $u_1$ with a continuous derivative and satisfies (3.6).*

**Proof.** We only prove the non-trivial parts b) and c).

**ad b)** The fact that $\tau$-homogeneity implies (3.6) is well-known ("Euler's relation"). In order to see that (3.6) implies $\tau$-homogeneity for $r$ fix $u \in U$ and set

$$(3.7) \qquad \phi_u(t) \quad \overset{\text{def}}{=} \quad r(t\,u), \quad t > 0\,.$$

Then $\phi_u$ is differentiable and

$$(3.8) \qquad \phi_u'(t) \quad = \quad \sum_{i=1}^{d} u_i\, \frac{\partial r}{\partial u_i}(t\,u), \quad t > 0\,.$$

Now, observe that (3.6) with (3.7) and (3.8) implies

$$\phi_u'(t) \quad = \quad \tau\, \frac{\phi_u(t)}{t}\,, \quad t > 0\,.$$

This ordinary differential equation in $t$ has a unique solution with initial value $\phi_u(1) = r(u)$, namely $\phi_u(t) = t^\tau\, r(u)$. This completes the proof of b).

**ad c)** Fix any $u \in U$ with $u_1 \neq 0$ and define

$$\zeta_u(t) \quad \overset{\text{def}}{=} \quad r(u_1, \frac{u_2}{t}, \ldots, \frac{u_d}{t}), \quad t > 0\,.$$

Then we have

$$(3.9) \qquad \zeta_u'(1) \quad = \quad - \sum_{j=2}^{d} u_j\, \frac{\partial r}{\partial u_j}(u)\,.$$

Consider now $\xi_u(t) \overset{\text{def}}{=} t^\tau\, \zeta_u(t),\ t > 0$. In this case, we have by (3.9)

$$\xi_u'(1) \quad = \quad \tau\, \zeta_u(1) + \zeta_u'(1) \quad = \quad \tau\, r(u) - \sum_{j=2}^{d} u_j\, \frac{\partial r}{\partial u_j}(u)\,.$$

The $\tau$-homogeneity implies $\xi_u(t) = r(t\,u_1, u_2, \ldots, u_d)$ and hence

$$u_1^{-1} \left( \tau\, r(u) - \sum_{j=2}^{d} u_j\, \frac{\partial r}{\partial u_j}(u) \right) \quad = \quad u_1^{-1} \xi_u'(1) \quad = \quad \frac{\partial r}{\partial u_1}(u)\,. \qquad \square$$

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# 4   Risk contributions and differentiable risk measures

Equation (3.6) is appealing because it suggests a natural way to apportion the portfolio risk to the single assets while simultaneously respecting their weights. There are good reasons for such an apportionment; see [22] for some of them. The most important might be risk adjusted performance measurement. In this section we show that careful assignment of risk contributions of the assets can be useful in optimizing performance measured as ratio of expected cash flow and economic capital. On the contrary, a thoughtless assignment may result in rather misleading hints concerning the portfolio management.

As with the notions of risk and economic capital we do not need any formal definition for *risk contribution*. Finding meaningful risk contributions corresponds to deciding from which vector field $a = (a_1, \ldots, a_d) : U \to \mathbb{R}^d$ most information can be inferred about a certain function $r : U \to \mathbb{R}$, the risk measure. In a differentiable context the answer seems clear: from the gradient of $r$.

Nonetheless, examining the problem more closely is instructive. We begin by defining the return function corresponding to a risk measure seen as an ordinary function.

**Definition 4.1**
*Let $\emptyset \neq U$ be a set in $\mathbb{R}^d$ and $r : U \to \mathbb{R}$ be some function on $U$. Fix any $m \in \mathbb{R}^d$. Then the function $g = g_{r,m} : \{u \in U \ : \ r(u) \neq m'u\} \to \mathbb{R}$, defined by*

$$(4.1) \qquad\qquad g(u) \quad \overset{\text{def}}{=} \quad \frac{m'u}{r(u) - m'u},$$

*is called* return function *for $r$.* □

As we see the economic capital as a reserve to compensate unexpected losses in the future it should be discounted with some factor when the portfolio return is calculated. The factor should depend on the length of the time interval under consideration and the risk-free interest rate. We don't care about this factor because we are not primarily interested in absolute performance but in performance relative to those of other portfolios or assets. If the economic capital $r(u^{(i)}) - m'u^{(i)}$ of portfolios $u^{(i)}, i = 1, 2$, is positive then it is clear that the performance of $u^{(1)}$ is better than that of $u^{(2)}$ if and only if $g(u^{(1)}) > g(u^{(2)})$. But we also allow negative values for the economic capital. This may be reasonable when considering a portfolio of guarantees or derivatives which are hold in order to reduce economic capital.

Observe that the case of opposite signs in the denominator and the numerator resp. of the quotient in (4.1) is unrealistic. On the one hand, the case of a positive numerator and a negative denominator means that someone gives us a present of a guarantee and even pays for being allowed to do so. On the other hand, the case of a negative numerator and a positive denominator means that we are so kind to pay for being allowed to bear someone else's risk.

More interesting is the case where both the denominator and the numerator in (4.1) are negative. In this case $g(u)$ depicts the profit of a counterparty and should therefore – from the investor's point of view – be hold as small as possible.

Keep the meanings of the signs in (4.1) in mind when interpreting the following definition. It translates the postulate that a risk contribution should give the right signals for portfolio management into a mathematical formulation.

**Definition 4.2**

*Let $\emptyset \neq U$ be a set in $\mathbb{R}^d$ and $r : U \to \mathbb{R}$ be some function on $U$.*
*A vector field $a = (a_1, \ldots, a_d) : U \to \mathbb{R}^d$ is called* suitable for performance measurement *with $r$ if it satisfies the following two conditions:*

(i) *For all $m \in \mathbb{R}^d$, $u \in U$ with $r(u) \neq m'u$ and $i \in N_d$ the inequality*

$$(4.2) \qquad\qquad m_i\, r(u) \quad > \quad a_i(u)\, m'u$$

*implies that there is an $\epsilon > 0$ such that for all $t \in (0, \epsilon)$ we have*

$$(4.3) \qquad\quad g_{r,m}(-t\, e^{(i)} + u) \quad < \quad g_{r,m}(u) \quad < \quad g_{r,m}(t\, e^{(i)} + u)\,.$$

(ii) *For all $m \in \mathbb{R}^d$, $u \in U$ with $r(u) \neq m'u$ and $i \in N_d$ the inequality*

$$(4.4) \qquad\qquad m_i\, r(u) \quad < \quad a_i(u)\, m'u$$

*implies that there is an $\epsilon > 0$ such that for all $t \in (0, \epsilon)$ we have*

$$(4.5) \qquad g_{r,m}(-t\, e^{(i)} + u) \quad > \quad g_{r,m}(u) \quad > \quad g_{r,m}(t\, e^{(i)} + u)\,. \qquad \square$$

**Remark 4.3**

(i) The quantity $a_i(u), i \in N_d$, may be regarded as the risk contribution of one unit or one piece of asset $i$ or as *normalized* risk contribution of asset $i$.

(ii) Evidently, (4.2) is equivalent to

$$(4.6) \qquad\qquad m_i\, (r(u) - m'u) \quad > \quad (a_i(u) - m_i)\, m'u\,,$$

and similarly for (4.4). Inequality (4.6) indicates the relation between the portfolio return $g(u)$ and the return $\frac{m_i}{a_i(u) - m_i}$ of asset $i$ as part of the portfolio which ensures that the portfolio return will increase when the weight of asset $i$ in the portfolio is increased.

(iii) We will see in Proposition 4.6 that suitableness for performance measurement as in definition 4.2 often implies a similar property for subportfolios consisting of more than one asset.

The following result shows that for a "smooth" function the only vector field which is suitable for performance measurement with the function is the gradient of the function.

**Theorem 4.4**

*Let $\emptyset \neq U \subset \mathbb{R}^d$ be an open set and $r : U \to \mathbb{R}$ be a function that is partially differentiable in $U$ with continuous derivatives. Let $a = (a_1, \ldots, a_d) : U \to \mathbb{R}^d$ be a continuous vector field.*

*Then $a$ is suitable for performance measurement with $r$ if and only if*

$$(4.7) \qquad a_i(u) \quad = \quad \frac{\partial r}{\partial u_i}(u), \quad i = 1, \ldots, d, \ u \in U.$$

**Proof.** Observe that for $u \in U$ with $r(u) \neq m'u$, $m \in \mathbb{R}^d$, and $i = 1, \ldots, d$ we get

$$(4.8) \qquad \frac{\partial g_{r,m}}{\partial u_i}(u, a) \quad =$$
$$(r(u) - m'u)^{-2} \left( m_i\, r(u) - a_i(u)\, m'u + \left( a_i(u) - \frac{\partial r}{\partial u_i}(u, a) \right) m'u \right).$$

If (4.7) is satisfied then the suitableness for performance measurement follows immediately from (4.8).

For the necessity of (4.7) fix any $i \in N_d$ and note that by continuity we only need to show (4.7) for $u \in U$ such that $u_i \neq 0$ and $u_j \neq 0$ for some $j \neq i$. Now, the proof is simple but requires some care for several special cases. These cases are:

(i) $a_i(u) \neq 0$, $r(u) \neq 0$, $r(u) \neq u_i\, a_i(u)$,

(ii) $a_i(u) \neq 0$, $r(u) \neq 0$, $r(u) = u_i\, a_i(u)$,

(iii) $a_i(u) = 0$, $r(u) \neq 0$,

(iv) $r(u) = 0$, each neighbourhood of $u$ contains some $v \in U$ such that $r(u) \neq 0$,

(v) $r(v) = 0$ for all $v$ in some neighbourhood of $u$.

We will only give a proof for case (i) because the proofs for (ii) and (iii) are almost identical, (iv) follows by continuity and (v) is trivial.

Choose any $j \in N_d \backslash \{i\}$ with $u_j \neq 0$ and define $m(t) \in \mathbb{R}^d$ by

$$m_i(t) \quad \overset{\text{def}}{=} \quad 1,$$
$$m_j(t) \quad \overset{\text{def}}{=} \quad \frac{t}{u_j}\left( \frac{r(u)}{a_i(u)} - u_i \right), \quad \text{and}$$
$$m_l(t) \quad \overset{\text{def}}{=} \quad 0 \quad \text{for } l \neq i, j.$$

Then

$$m(t)'u \quad = \quad t\,\frac{r(u)}{a_i(u)} + (1-t)\,u_i \quad \text{and}$$

$$m_i(t)\,r(u) - a_i(u)\,m(t)'u \quad = \quad (1-t)\,(r(u) - u_i\,a_i(u))\,.$$

Hence by suitableness and (4.8) we can choose sequences $(t_k)$ and $(s_k)$ with $t_k \to 1$ and $s_k \to 1$ such that for all $k \in \mathbb{N}$ we have $m(s_k)'u \neq r(u) \neq m(t_k)'u$ as well as

$$(1-t_k)\,(r(u) - u_i\,a_i(u)) \; + \; \left( a_i(u) - \frac{\partial r}{\partial u_i}(u) \right)\left( t_k\,\frac{r(u)}{a_i(u)} + (1-t_k)\,u_i \right) \;\; \geq \;\; 0 \quad \text{and}$$

$$(1-s_k)\,(r(u) - u_i\,a_i(u)) \; + \; \left( a_i(u) - \frac{\partial r}{\partial u_i}(u) \right)\left( s_k\,\frac{r(u)}{a_i(u)} + (1-s_k)\,u_i \right) \;\; \leq \;\; 0\,.$$

Now $k \to \infty$ yields (4.7). $\qquad\square$

In [8] (sec. 5) the author shows by arguments from game theory that in case of a 1-homogeneous risk measure its gradient is the only "allocation principle" that fulfills some "coherence" postulates.

By Theorem 4.4 we know that, if a risk measure is smooth, we should use its partial derivatives as risk contributions of the assets in the portfolio. Otherwise we run the risk of receiving misleading informations about the profitability of the assets. Let us review the concept of marginal risk, known from literature, under this point of view.

**Example 4.5**

Let $r : U \to \mathbb{R}$ be any risk measure for some portfolio with assets $1, \ldots, d$. Some authors (cf. [18] ch. 6 or [15]) suggest the application of the so-called "marginal risk" for determining the capital required by an individual business or asset. Formally, the marginal risk $r_i$ of asset $i$, $i = 1, \ldots, d$, is defined by

$$(4.9) \qquad\qquad r_i(u) \quad \overset{\text{def}}{=} \quad r(u) - r(u - u_i\,e^{(i)}), \quad u \in \mathbb{R}^d\,,$$

i.e. by the difference of the portfolio risk with asset $i$ and the portfolio risk without asset $i$. Setting for $i = 1, \ldots, d$

$$(4.10) \qquad\qquad a_i(u) \quad \overset{\text{def}}{=} \quad \frac{r_i(u)}{u_i}, \quad u \in \mathbb{R}^d,\; u_i \neq 0\,,$$

creates a vector field $a = (a_1, \ldots, a_d)$ measuring normalized risk contributions of the assets in the sense of Remark 4.3 (i) (see also [11]).

If $r$ is differentiable then, in general, $a$ will not be identical with the gradient of $r$. To see this note that by the mean value theorem for $u \in \mathbb{R}^d$ there are numbers $\theta_i(u) \in [0,1]$, $i = 1, \ldots, d$, such that

$$(4.11) \qquad\qquad r_i(u) \quad = \quad u_i\,\frac{\partial r}{\partial u_i}(u - \theta_i(u)\,u_i\,e^{(i)})\,.$$

By (4.11) and (4.10) in general we have

$$a_i(u) \quad \neq \quad \frac{\partial r}{\partial u_i}(u)$$

and hence by Theorem 4.4 $a$ will not be suitable for performance measurement with $r$. If $r$ is also 1-homogeneous then by Proposition 3.5 b) it has the nice feature that

$$r(u) \quad = \quad \sum_{i=1}^{d} u_i \frac{\partial r}{\partial u_i}(u).$$

Equation (4.11) now reveals that the equality $\sum_{i=1}^{d} r_i(u) = r(u)$ is unlikely.  □

Observe that Theorem 4.4 suggests a more appropriate way for calculating a meaningful marginal risk of asset $i$: simply use the difference quotient

$$(4.12) \qquad\qquad h^{-1}\left(r(u + h\,e^{(i)}) - r(u)\right) \quad \approx \quad \frac{\partial r}{\partial u_i}(u)$$

with some suitable small $h \neq 0$.

The notion of suitableness for performance management is based on the consideration of single assets. The following proposition says that the gradient of a risk measure also provides useful information about the profitability of subportfolios consisting of more than one asset.

For a unit vector $\nu \in \mathbb{R}^d$ (i.e. $\nu'\nu = 1$) denote by $\frac{\partial \phi}{\partial \nu}$ the derivative

$$\frac{\partial \phi}{\partial \nu}(u) \quad = \quad \sum_{i=1}^{d} \nu_i \frac{\partial \phi}{\partial u_i}(u)$$

of the function $\phi$ in direction $\nu$.

See Remark 4.3 (ii) for the interpretation of (4.13) and (4.14).

**Proposition 4.6**

*Let $\emptyset \neq U \subset \mathbb{R}^d$ be an open set and $r : U \to \mathbb{R}$ any function which is partially differentiable in $U$ with continuous derivatives. Let $\nu \in \mathbb{R}^d$ be an arbitrary unit vector.*

*(i) For all $m \in \mathbb{R}^d$, $u \in U$ with $r(u) \neq m'u$ and*

$$(4.13) \qquad\qquad m'\nu\, r(u) \quad > \quad m'u \frac{\partial r}{\partial \nu}(u)$$

*there is an $\epsilon > 0$ such that the mapping*

$$t \mapsto g_{r,m}(u + t\,\nu), \ (-\epsilon, \epsilon) \to \mathbb{R}$$

*is strictly increasing.*

*(ii) For all $m \in \mathbb{R}^d$, $u \in U$ with $r(u) \neq m'u$ and*

$$(4.14) \qquad\qquad m'\nu\, r(u) \quad < \quad m'u \frac{\partial r}{\partial \nu}(u)$$

*there is an $\epsilon > 0$ such that the mapping*

$$t \mapsto g_{r,m}(u + t\,\nu), \ (-\epsilon, \epsilon) \to \mathbb{R}$$

*is strictly decreasing.*

**Proof.** Proposition 4.6 is an immediate consequence of the following equality:

$$\frac{d\,g}{d\,t}(u + t\,\nu)\bigg|_{t=0} \quad = \quad (r(u) - m'u)^{-2}\left(m'\nu\, r(u) - m'u\,\frac{\partial r}{\partial \nu}(u)\right). \qquad \square$$

From Definition 4.2 and Theorem 4.4 the reader will expect that the returns of all sub-portfolios are equal if a portfolio is optimal in the sense of a maximal return $g_{r,m}(u)$. Formally, this is stated in the following theorem.

**Theorem 4.7**

*Let $\emptyset \neq U \subset \mathbb{R}^d$ be an open set and $r : U \to \mathbb{R}$ a function that is partially differentiable in $U$ with continuous derivatives.*
*Let $\emptyset \neq I \subset N_d$, $m \in \mathbb{R}^d$ and $v \in U$ with $r(v) \neq m'v$ be fixed.*
*Assume that there is an $\epsilon > 0$ such that for all $u \in U$ with $\mid u_i - v_i \mid < \epsilon$ for $i \in I$ and $u_i = v_i$ for $i \notin I$ we have $r(u) \neq m'u$ and*

$$(4.15) \qquad\qquad\qquad g_{r,m}(v) \quad \geq \quad g_{r,m}(u).$$

*Then*

$$(4.16) \qquad\qquad\qquad m_i\, r(v) \quad = \quad m'v\, \frac{\partial r}{\partial u_i}(v), \quad i \in I.$$

*If, moreover, $r$ is 1-homogeneous and $I \neq N_d$ then we also have*

$$(4.17) \qquad\qquad \left(\sum_{j\notin I} m_j\, v_j\right) r(v) \quad = \quad m'v \sum_{j\notin I} v_j\, \frac{\partial r}{\partial u_j}(v).$$

**Proof.**
(4.16) is obvious from equation (4.8) in the proof of Theorem 4.4.
Assume now that $r$ is 1-homogeneous. Then by Proposition 3.5 b)

$$\sum_{j\notin I} v_j\, \frac{\partial r}{\partial u_j}(v) \quad = \quad r(v) - \sum_{j\in I} v_j\, \frac{\partial r}{\partial u_j}(v).$$

Together with (4.16) this implies

$$
\begin{aligned}
\left(\sum_{j\notin I} m_j\, v_j\right) r(v) \quad &= \quad r(v)\, m'v - \sum_{j\in I} m_j\, r(v)\, v_j \\
&= \quad m'v\left(r(v) - \sum_{j\in I} v_j\, \frac{\partial r}{\partial u_j}(v)\right) \\
&= \quad m'v \sum_{j\notin I} v_j\, \frac{\partial r}{\partial u_j}(v). \qquad \square
\end{aligned}
$$

**Remark 4.8**
If $m_i \neq \frac{\partial r}{\partial u_i}(v)$ then (4.16) says that the return

$$\frac{m_i}{\frac{\partial r}{\partial u_i}(v) - m_i}$$

of asset $i$ equals the optimal portfolio return $g_{r,m}(v)$.

Similarly, if $\sum_{j \notin I} m_j v_j \neq \sum_{j \notin I} v_j \frac{\partial r}{\partial u_j}(v)$ then (4.17) states that the subportfolio return

$$\frac{\sum_{j \notin I} m_j v_j}{\sum_{j \notin I} v_j \frac{\partial r}{\partial u_j}(v) - \sum_{j \notin I} m_j v_j}$$

equals the portfolio return $g_{r,m}(v)$ as well. $\qquad\square$

# 5 Examples of risk contributions

In this section we compute the derivatives of the risk measures introduced as examples in Section 3. The resulting risk contributions have appealing interpretations as predictors of the asset cash flows given a worst case scenario for the portfolio cash flow. For the VaR the risk contributions obtained by differentiation differ from the covariance based contributions that are widely used in practice.

## 5.1 Covariance based risk contributions

Let us briefly recall the notion *best linear predictor*. Assume that $Y$ and $Z$ are square-integrable real random variables on the same probability space. If $\mathrm{var}(Z) > 0$ then we can compute the projection $\pi_Z(z, Y)$ of $Y - \mathrm{E}\,[\,Y\,]$ onto the linear space spanned by $Z - \mathrm{E}\,[\,Z\,]$ via

$$(5.1) \qquad \pi_Z(z, Y) \quad = \quad \frac{\mathrm{cov}(Y, Z)}{\mathrm{var}(Z)}\, z, \quad z \in \mathbb{R}\,.$$

$\pi_Z(z, Y)$ is the best linear predictor of $Y - \mathrm{E}\,[\,Y\,]$ given $Z - \mathrm{E}\,[\,Z\,] = z$ in the sense that the random variable $\pi_Z(Z - \mathrm{E}\,[\,Z\,], Y)$ minimizes the $L_2$-distance between $Y - \mathrm{E}\,[\,Y\,]$ and the linear space spanned by $Z - \mathrm{E}\,[\,Z\,]$. Choosing a value for $z$ corresponds to defining a worst-case scenario for the portfolio cash flow. We first consider the case $z = c\,\sqrt{\mathrm{var}(Z)}$ in (5.1).

**Example 5.1** (Continuation of Example 3.1)

Define $U \subset \mathbb{R}^d$ by

$$(5.2) \qquad\qquad U \quad \overset{\mathrm{def}}{=} \quad \left\{u \in \mathbb{R}^d \mid \mathrm{var}(Z(u)) > 0\right\}$$

and suppose $U \neq \emptyset$. Then $U$ is a homogeneous open set. For $u \in U$ define the vector field $a = (a_1, \ldots, a_d) : U \to \mathbb{R}^d$ by

$$(5.3) \qquad a_i(u) \quad \overset{\mathrm{def}}{=} \quad \pi_{Z(u)}(r(u), X_i) \quad = \quad c\,\frac{\mathrm{cov}(X_i, Z(u))}{\sqrt{\mathrm{var}(Z(u))}}, \quad i = 1, \ldots, d\,.$$

Thus $a_i(u)$ is the best linear predictor of the cash flow fluctuation of asset $i$ given that the portfolio fluctuation is just the risk $r(u)$ defined in Example 3.1. In $u \in U$ we have

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for $i = 1, \ldots, d$

$$2\, r(u) \frac{\partial r}{\partial u_i}(u) = \frac{\partial r^2}{\partial u_i}(u)$$

$$= c^2 \frac{\partial}{\partial u_i} \left( \sum_{j=1}^{d} \sum_{l=1}^{d} u_j\, u_l \operatorname{cov}(X_j, X_l) \right)$$

$$= 2\, c^2 \operatorname{cov}(X_i, Z(u))$$

and hence
$$(5.4) \qquad \frac{\partial r}{\partial u_i}(u) = c^2 \frac{\operatorname{cov}(X_i, Z(u))}{r(u)} = a_i(u)\,.$$

By Theorem 4.4 $a$ is thus suitable for performance measurement with $r$. Moreover, since $r$ is 1-homogeneous we know from Proposition 3.5 b) without computation that

$$r(u) = \sum_{i=1}^{d} u_i\, a_i(u), \quad u \in U. \qquad \square$$

Another appealing choice for the value of $z$ in (5.1) is $z = Q_\alpha(u)$. This leads us to the situation of Example 3.2.

**Example 5.2** (Continuation of Example 3.2)
Define again $U \subset \mathbb{R}^d$ by (5.2) and suppose $U \neq \emptyset$. For $u \in U$ define analogously to Example 5.1 the vector field $a = (a_1, \ldots, a_d) : U \to \mathbb{R}^d$ by

$$(5.5) \qquad a_i(u) \overset{\text{def}}{=} \pi_{Z(u)}(r(u), X_i) = \frac{\operatorname{cov}(X_i, Z(u))}{\operatorname{var}(Z(u))} Q_\alpha(u), \quad i = 1, \ldots, d\,.$$

Then we have again $r(u) = \sum_{i=1}^{d} u_i\, a_i(u), u \in U$. This method for determining the contributions of the assets is proposed for instance in Section 6.1 of [19] or in Appendix A13 of [7]. We will see in the next subsection that in general we have $a_i \neq \frac{\partial r}{\partial u_i}$, and hence $a$ is not suitable for performance measurement with $r$. $\qquad \square$

Observe that in case of an elliptically (and in particular of a normally) distributed random vector $(X_1, \ldots, X_d)$ equations (5.3) and (5.5) lead to the same result when the constant $c$ is chosen as the $\alpha$-quantile of the standardized univariate marginal distribution (Theorem 1 in [10]). If the distribution of $(X_1, \ldots, X_d)$ is not an elliptical distribution, the $a_i$ and the $\frac{\partial r}{\partial u_i}$ in Example 5.2 can considerably differ. In particular, this may be the case in credit portfolios (cf. [15], Sec. 1.1.2).

## 5.2 Quantile based risk contributions

In this subsection we will compute the risk contributions that are associated with the VaR risk measure from Example 3.2 via differentiation. However, in general the quantile function $Q_\alpha(u)$ from (3.2) will not be differentiable in $u$. In order to guarantee that

differentiation is possible we have to impose some technical assumptions on the joint distribution of the fluctuation vector $(X_1, \ldots, X_d)$. The most important one among these could roughly stated as: at least one among the fluctuations $X_i$ must have a continuous density.

**Assumption (S)**

For fixed $\alpha \in (0,1)$, we say that an $\mathbb{R}^d$-valued random vector $(X_1, \ldots, X_d)$ satisfies Assumption (S) if $d \geq 2$ and the conditional distribution of $X_1$ given $(X_2, \ldots, X_d)$ has a density

$$\phi : \mathbb{R} \times \mathbb{R}^{d-1} \to [0, \infty), \quad (t, x_2, \ldots, x_d) \mapsto \phi(t, x_2, \ldots, x_d)$$

which satisfies the following four conditions:

(i) For fixed $x_2, \ldots, x_d$ the function $t \mapsto \phi(t, x_2, \ldots, x_d)$ is continuous in $t$.

(ii) The mapping

$$(t, u) \mapsto \mathrm{E}\left[\phi\left(u_1^{-1}\left(t - \sum_{j=2}^d u_j X_j\right), X_2, \ldots, X_d\right)\right], \quad \mathbb{R} \times \mathbb{R}\backslash\{0\} \times \mathbb{R}^{d-1} \to [0, \infty)$$

is finite-valued and continuous.

(iii) For each $u \in \mathbb{R}\backslash\{0\} \times \mathbb{R}^{d-1}$

$$0 \quad < \quad \mathrm{E}\left[\phi\left(u_1^{-1}\left(Q_\alpha(u) - \sum_{j=2}^d u_j X_j\right), X_2, \ldots, X_d\right)\right],$$

with $Q_\alpha(u)$ defined by (3.2).

(iv) For each $i = 2, \ldots, d$ the mapping

$$(t, u) \mapsto \mathrm{E}\left[X_i \, \phi\left(u_1^{-1}\left(t - \sum_{j=2}^d u_j X_j\right), X_2, \ldots, X_d\right)\right], \quad \mathbb{R} \times \mathbb{R}\backslash\{0\} \times \mathbb{R}^{d-1} \to \mathbb{R}$$

is finite-valued and continuous. $\qquad\qquad\square$

Note that (i) in general implies neither (ii) nor (iv). Furthermore, (ii) and (iv) may be valid even if the components of the random vector $(X_1, \ldots, X_d)$ do not have finite expectations. Before turning to the next result let us just present some situations in which Assumption (S) is satisfied:

1) $(X_1, \ldots, X_d)$ is normally distributed and its covariance matrix has full rank.

2) $(X_1, \ldots, X_d)$ and $\phi$ satisfy (i) and (iii) resp. and for each $(s, v) \in \mathbb{R} \times \mathbb{R}\backslash\{0\} \times \mathbb{R}^{d-1}$ there is some neighbourhood $V$ such that the random fields

$$\left(\phi\left(u_1^{-1}(t - \sum_{j=2}^d u_j X_j), X_2, \ldots, X_d\right)\right)_{(t,u)\in V}$$

and for $i = 2, \ldots, d$

$$\left(X_i \, \phi\left(u_1^{-1}(t - \sum_{j=2}^d u_j X_j), X_2, \ldots, X_d\right)\right)_{(t,u)\in V}$$

are uniformly integrable.

3) $\mathrm{E}\,[\,|\,X_i\,|\,] < \infty$, $i = 2, \ldots, d$, and $\phi$ is bounded and and satisfies (i) and (iii). .

4) $\mathrm{E}\,[\,|\,X_i\,|\,] < \infty$, $i = 2, \ldots, d$. $X_1$ and $(X_2, \ldots, X_d)$ are independent. $X_1$ has a continuous density $f$ such that

$$0 \quad < \quad \mathrm{E}\left[\, f\left(u_1^{-1}\left(Q_\alpha(u) - \sum_{j=2}^d u_j\, X_j\right)\right)\right].$$

5) There is a finite set $M \subset \mathbb{R}^{d-1}$ such that

$$\mathrm{P}\,[\,(X_2, \ldots, X_d) \in M\,] = 1\,,$$

and (i) and (iii) are satisfied.

Note that 3) is a special case of 2) and that 4) and 5) resp. are special cases of 3). 4) shows that $Q_\alpha(u)$ can be forced to be differentiable by disturbing the portfolio cash flow fluctuation $Z(u)$ with some small independent noise.

**Lemma 5.3**

*For some given $\alpha \in (0,1)$, let $(X_1, \ldots, X_d)$ be an $\mathbb{R}^d$-valued random vector satisfying Assumption (S). Set $U \overset{\text{def}}{=} \mathbb{R}\backslash\{0\} \times \mathbb{R}^{d-1}$ and define the random field $(Z(u))_{u \in U}$ by*

$$Z(u) \quad \overset{\text{def}}{=} \quad \sum_{i=1}^d u_i\, X_i\,, \quad u \in U\,.$$

*Then the function $Q_\alpha : U \to \mathbb{R}$ with*

$$Q_\alpha(u) \quad \overset{\text{def}}{=} \quad \inf\{z \in \mathbb{R} : \mathrm{P}\,[\,Z(u) \le z\,] \ge \alpha\}, \quad u \in U\,,$$

*is partially differentiable in $U$ with continuous derivatives*

(5.6) $$\frac{\partial Q_\alpha}{\partial u_1}(u) =$$

$$u_1^{-1}\left(Q_\alpha(u) - \frac{\mathrm{E}\left[\left(\sum_{j=2}^d u_j\, X_j\right)\phi\left(u_1^{-1}(Q_\alpha(u) - \sum_{j=2}^d u_j\, X_j), X_2, \ldots, X_d\right)\right]}{\mathrm{E}\left[\phi\left(u_1^{-1}(Q_\alpha(u) - \sum_{j=2}^d u_j\, X_j), X_2, \ldots, X_d\right)\right]}\right)$$

*and*

(5.7) $$\frac{\partial Q_\alpha}{\partial u_i}(u) = \frac{\mathrm{E}\left[X_i\,\phi\left(u_1^{-1}(Q_\alpha(u) - \sum_{j=2}^d u_j\, X_j), X_2, \ldots, X_d\right)\right]}{\mathrm{E}\left[\phi\left(u_1^{-1}(Q_\alpha(u) - \sum_{j=2}^d u_j\, X_j), X_2, \ldots, X_d\right)\right]}, i = 2, \ldots, d.$$

**Proof.** We only need to give the proof for $u_1 > 0$ since we then obtain the formulas for $u_1 < 0$ by considering the random vector $(X_1', \ldots, X_d') = (-X_1, X_2, \ldots, X_d)$.
Thus, define $F : \mathbb{R} \times (0, \infty) \times \mathbb{R}^{d-1} \to \mathbb{R}$ by

(5.8) $$F(y, u) \quad \overset{\text{def}}{=} \quad \mathrm{P}\,[\,Z(u) \le y\,]\,.$$

By setting

$$G(y, u, x_2, \ldots, x_d) \quad \overset{\text{def}}{=} \quad \int_{-\infty}^{u_1^{-1}(y - \sum_{j=2}^d u_j\, x_j)} \phi(t, x_2, \ldots, x_d)\, dt$$

$$= \quad \mathrm{P}\left[\, Z(u) \le y \mid X_2 = x_2, \ldots, X_d = x_d \,\right]$$

we can write $F(y, u)$ as

$$(5.9) \qquad\qquad F(y, u) \quad = \quad \mathrm{E}\left[\, G(y, u, X_2, \ldots, X_d) \,\right].$$

In a first step we want to show that $F(y, u)$ is continuously differentiable and that its derivatives may be computed by changing the order of integration and differentiation on the right-hand side of (5.9). On this behalf we evoke Theorem A.(9.1) from [9] and verify its conditions (i) – (iv). (i) is clear as $F(y, u)$ is a probability.

Observe that the function $G$ is partially differentiable in $y$ and $u_i, i = 1, \ldots, d$. For the derivatives we calculate

$$\frac{\partial G}{\partial y}(y, u, x_2, \ldots, x_d) \quad = \quad \tfrac{1}{u_1} \phi\left(u_1^{-1}(y - \sum_{j=2}^d u_j\, x_j), x_2, \ldots, x_d\right),$$

$$\frac{\partial G}{\partial u_1}(y, u, x_2, \ldots, x_d) \quad = \quad -u_1^{-2}\left(y - \sum_{j=2}^d u_j\, x_j\right) \phi\left(u_1^{-1}(y - \sum_{j=2}^d u_j\, x_j), x_2, \ldots, x_d\right), \text{ and}$$

$$\frac{\partial G}{\partial u_i}(y, u, x_2, \ldots, x_d) \quad = \quad -\tfrac{x_i}{u_1} \phi\left(u_1^{-1}(y - \sum_{j=2}^d u_j\, x_j), x_2, \ldots, x_d\right), \quad i = 2, \ldots, d.$$

By Assumption (S) (i) for fixed $x_2, \ldots, x_d$ these derivatives are continuous in $(y, u)$, giving (ii) of Theorem A.(9.1). Moreover, by Assumption (S) (ii) and (iv) the expressions

$$\mathrm{E}\left[\frac{\partial G}{\partial u_i}(y, u, X_2, \ldots, X_d)\right], \quad i = 1, \ldots, d, \quad \text{and} \quad \mathrm{E}\left[\frac{\partial G}{\partial y}(y, u, X_2, \ldots, X_d)\right]$$

are also continuous in $(y, u)$. This is (iii) of Theorem A.(9.1). Condition (iv) of A.(9.1) follows with some computations from the definition of conditional densities. Hence we know that $F(y, u)$ is continuously partially differentiable with

$$(5.10) \qquad \frac{\partial F}{\partial u_i}(y, u) \quad = \quad \mathrm{E}\left[\frac{\partial G}{\partial u_i}(y, u, X_2, \ldots, X_d)\right], \quad i = 1, \ldots, d, \quad \text{and}$$

$$(5.11) \qquad \frac{\partial F}{\partial y}(y, u) \quad = \quad \mathrm{E}\left[\frac{\partial G}{\partial y}(y, u, X_2, \ldots, X_d)\right].$$

(5.11) implies in particular $\mathrm{P}\left[\, Z(u) = y \,\right] = 0$ for all $y \in \mathbb{R}$ and $u \in (0, \infty) \times \mathbb{R}^{d-1}$. From this we obtain

$$F(Q_\alpha(u), u) \quad = \quad \alpha$$

for all $u \in (0, \infty) \times \mathbb{R}^{d-1}$. Hence by Assumption (S) (iii) and the theorem of implicit functions we may conclude that $Q_\alpha(u)$ is continuously partially differentiable. Its derivatives

can be deduced from (5.10) and (5.11) by the formula

$$\frac{\partial Q_\alpha}{\partial u_i}(u) = -\left(\frac{\partial F}{\partial y}(Q_\alpha(u), u)\right)^{-1} \frac{\partial F}{\partial u_i}(Q_\alpha(u), u), \quad i = 1, \dots, d. \qquad \square$$

**Remark 5.4**

Equations (5.6) and (5.7) allow an interesting interpretation. Fix $u \in \mathbb{R}\backslash\{0\} \times \mathbb{R}^d$ and set for $z \in \mathbb{R}$

$$g_u(z) \quad \overset{\text{def}}{=} \quad \mathrm{E}\left[\phi\left(u_1^{-1}(z - \textstyle\sum_{j=2}^d u_j\, X_j), X_2, \dots, X_d\right)\right].$$

By (5.11) we know that $\frac{g_u}{|u_1|}$ is a continuous density of the random variable $Z(u)$. It is not hard to see that then for $i = 2, \dots, d$ the functions $h_u^{(i)}$ with

$$h_u^{(i)}(z) \quad \overset{\text{def}}{=} \quad \begin{cases} 0, & \text{if } g_u(z) = 0 \\ g_u(z)^{-1}\mathrm{E}\left[X_i\, \phi\left(u_1^{-1}(z - \textstyle\sum_{j=2}^d u_j\, X_j), X_2, \dots, X_d\right)\right], & \text{otherwise,} \end{cases}$$

provide versions of $\mathrm{E}[X_i \mid Z(u) = \cdot]$, the conditional expectation of $X_i$ given $Z(u)$. Similarly we have

$$\mathrm{E}[X_1 \mid Z(u) = z] \quad = \quad u_1^{-1}\left(z - \textstyle\sum_{j=2}^d u_j\, h_u^{(j)}(z)\right).$$

Hence Lemma 5.3 says nothing else than

(5.12) $\qquad \dfrac{\partial Q_\alpha}{\partial u_i}(u) \quad = \quad \mathrm{E}[X_i \mid Z(u) = Q_\alpha(u)], \quad i = 1, \dots, d. \qquad \square$

Equation (5.12) has been presented in [13] without examination of the question whether $Q_\alpha$ is differentiable and in [12] for the case of $(X_1, \dots, X_d)$ with a joint density.

Recall that the conditional expectation of $X_i$ given $Z(u)$ essentially may be seen as the best predictor of $X_i$ by elements of the space $M \overset{\text{def}}{=} \{f(Z(u)) \mid f : \mathbb{R} \to \mathbb{R} \text{ measurable}\}$. As mentioned above the best linear predictor of $X_i$ given $Z(u)$ is the best predictor of $X_i$ by elements of the space $\{m\, Z(u) \mid m \in \mathbb{R}\} \subset M$.

We are now in a position to discuss Examples 3.2 and 5.2 again.

**Example 5.5** (Continuation of Example 3.2)

By Lemma 5.3 under Assumption (S) for $i = 1, \dots, d$ the mappings $b_i : \mathbb{R}\backslash\{0\} \times \mathbb{R}^{d-1} \to \mathbb{R}$ with

(5.13) $\qquad\qquad\qquad b_i(u) \quad \overset{\text{def}}{=} \quad \dfrac{\partial r}{\partial u_i}(u) \quad = \quad \dfrac{\partial Q_\alpha}{\partial u_i}(u)$

are well-defined. They provide a vector field of risk contributions $b = (b_1, \dots, b_d)$ which by Proposition 3.5 b) satisfies

$$r(u) \quad = \quad \sum_{i=1}^d u_i\, b_i(u), \quad u \in U,$$

and is suitable for performance measurement with $r$ by Theorem 4.4. By equation (5.12) we see that in general the vector fields $a$ from Example 5.2 and $b$ are not identical unless the random vector $(X_1, \dots, X_d)$ is elliptically distributed (cf. [10] Sec. 3.3). $\qquad \square$

## 5.3 Shortfall based risk contributions

As in the previous subsection for the quantile based risk we calculate here the risk contributions which are associated to the shortfall based risk (cf. Example 3.3) via differentiation. Again there is the problem that the quantile function in general might not be differentiable. Nevertheless, the following lemma and its corollary show that we may differentiate the shortfall under almost the same assumptions as those for the quantile.

For any set $A$ define $I(A, a) = I(A)$ by $I(A, a) \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } a \in A \\ 0, & \text{if } a \notin A. \end{cases}$

**Lemma 5.6**

Let $(X_1, \ldots, X_d)$ and $\alpha$ be as in Lemma 5.3 and assume

$$\mathrm{E}\left[\,|\,X_i\,|^n\,\right] < \infty, \quad i = 1, \ldots, d,$$

for some integer $n \geq 1$. Define $U$, $Z(u)$ and $Q_\alpha(u)$ as in Lemma 5.3 and set

$$T_{\alpha,n}(u) \quad \stackrel{\text{def}}{=} \quad \mathrm{E}\left[\,Z(u)^n \mid Z(u) \geq Q_\alpha(u)\,\right], \quad u \in U.$$

Then $T_{\alpha,n}$ on $U$ is continuous and partially differentiable in $u_i$, $i = 1, \ldots, d$, with continuous derivatives

$$(5.14) \qquad \frac{\partial T_{\alpha,n}}{\partial u_i}(u) = n\,\mathrm{E}\left[\,X_i\,Z(u)^{n-1} \mid Z(u) \geq Q_\alpha(u)\,\right], \quad i = 1, \ldots, d.$$

**Proof.** We may again assume $u_1 > 0$. Note first that under Assumption (S) for each $u \in U$ the distribution of $Z(u)$ is continuous and thus in particular we have

$$\mathrm{P}\left[\,Z(u) \geq Q_\alpha(u)\,\right] \quad = \quad 1 - \alpha \quad > \quad 0.$$

By Lemma 5.3 the quantile function $Q_\alpha$ is continuous. Hence by the representation

$$T_{\alpha,n}(u) \quad = \quad Q_\alpha(u)^n + (1-\alpha)^{-1}\mathrm{E}\left[\,(Z(u)^n - Q_\alpha(u)^n)\,I(Z(u) \geq Q_\alpha(u))\,\right]$$

and by $\mathrm{E}\left[\,|\,X_i\,|^n\,\right] < \infty$, $i = 1, \ldots, d$, the function $T_{\alpha,n}$ is continuous, too. Therefore, by Proposition 3.5 c), we only need to show (5.14) and the continuity of the derivatives for $i = 2, \ldots, d$.

Define $H(y, u) \stackrel{\text{def}}{=} \mathrm{P}\left[\,Z(u) > y\,\right]$ and note that

$$(5.15) \qquad n\int_y^\infty t^{n-1}\,H(t, u)\,dt + y^n\,\mathrm{P}\left[\,Z(u) > y\,\right] \quad = \quad \mathrm{E}\left[\,Z(u)^n\,I(Z(u) > y)\,\right].$$

Moreover, $H(y, u) = 1 - F(y, u)$ with $F(y, u)$ defined as in (5.8). Hence we know from the proof of Lemma 5.3 that $H(y, u)$ is continuous and partially differentiable in $u_i$ for $i = 2, \ldots, d$, with continuous derivatives

$$(5.16) \qquad \frac{\partial H}{\partial u_i}(y, u) \quad = \quad u_1^{-1}\mathrm{E}\left[\,X_i\,\phi\left(u_1^{-1}\left(y - \sum_{j=2}^d u_j\,X_j\right), X_2, \ldots, X_d\right)\right].$$

This representation for $\frac{\partial H}{\partial u_i}$ implies for $i = 2, \ldots, d$

$$\int\limits_y^\infty t^{n-1} \frac{\partial H}{\partial u_i}(t, u) \, dt$$

(5.17)
$$= \mathrm{E}\left[ X_i \int\limits_{u_1^{-1}(y - \sum_{j=2}^d u_j X_j)}^\infty \left( t\, u_1 + \sum_{j=2}^d u_j X_j \right)^{n-1} \phi(t, X_2, \ldots, X_d) \, dt \right]$$

$$= \mathrm{E}\left[ X_i \, Z(u)^{n-1} \, I(Z(u) \geq y) \right].$$

Equality (5.17) shows that the mapping

$$(y, u) \mapsto \int\limits_y^\infty t^{n-1} \frac{\partial H}{\partial u_i}(t, u) \, dt = \mathrm{E}\left[ X_i \, Z(u)^{n-1} \, I(Z(u) \geq y) \right]$$

is jointly continuous in $y$ and $u$. By (5.15) and (5.16), using Assumption (S) and the finiteness of the $n$-th absolute moment of $X_i$ for $i = 1, \ldots, d$, one now can verify that the conditions of Theorem A.(9.1) in [9] are satisfied. Hence we may change the order of integration and differentiation in $\frac{\partial}{\partial u_i} \int_y^\infty t^{n-1} H(t, u) \, dt$. This yields

(5.18)
$$\frac{\partial}{\partial u_i} \int\limits_y^\infty t^{n-1} H(t, u) \, dt = \mathrm{E}\left[ X_i \, Z(u)^{n-1} \, I(Z(u) \geq y) \right], \quad i = 2, \ldots, d.$$

By finiteness of $\int_y^\infty t^{n-1} H(t, u) \, dt$ and continuity of $t \mapsto H(t, u)$ we have

(5.19)
$$\frac{\partial}{\partial y} \int\limits_y^\infty t^{n-1} H(t, u) \, dt = -y^{n-1} \, \mathrm{P}\left[ Z(u) \geq y \right].$$

Since $\frac{\partial Q_\alpha}{\partial u_i}$ exists and is continuous by Lemma 5.3, from (5.15),(5.18) and (5.19) we deduce for $i = 2, \ldots, d$ that

$$\frac{\partial T_{\alpha,n}}{\partial u_i}(u) = \frac{n}{1-\alpha} \left( -(1-\alpha) \, Q_\alpha(u)^{n-1} \frac{\partial Q_\alpha}{\partial u_i}(u) + \mathrm{E}\left[ X_i \, Z(u)^{n-1} \, I(Z(u) \geq Q_\alpha(u)) \right] \right)$$

$$\qquad + n \, Q_\alpha(u)^{n-1} \frac{\partial Q_\alpha}{\partial u_i}(u)$$

$$= n \, \mathrm{E}\left[ X_i \, Z(u)^{n-1} \mid Z(u) \geq Q_\alpha(u) \right].$$

This is the desired result. $\qquad \square$

**Corollary 5.7**
*Let $(X_1, \ldots, X_d)$, $\alpha$ and $n$ be as in Lemma 5.6. Define also $Z(u)$, $Q_\alpha(u)$ and $T_{\alpha,n}$ as in Lemma 5.6 and set*

$$U_n \stackrel{\text{def}}{=} \begin{cases} \mathbb{R}\backslash\{0\} \times \mathbb{R}^{d-1}, & n = 1 \\ \left(\mathbb{R}\backslash\{0\} \times \mathbb{R}^{d-1}\right) \backslash \{u : T_{\alpha,n} = 0\}, & n \geq 2. \end{cases}$$

*Then the function $S_{\alpha,n} : U_n \to \mathbb{R}$, defined by*

$$S_{\alpha,n} \overset{\text{def}}{=} \sqrt[n]{T_{\alpha,n}(u)}, \quad u \in U_n,$$

*is continuous on $U_n$ and partially differentiable in $u_i, i = 1, \ldots, d$, with continuous derivatives*

(5.20) $$\frac{\partial S_{\alpha,n}}{\partial u_i}(u) = (S_{\alpha,n}(u))^{-(n-1)} \, \mathrm{E}\left[ X_i \, Z(u)^{n-1} \mid Z(u) \geq Q_\alpha(u) \right].$$

**Proof.** Obvious from Lemma 5.6. □

Lemma 5.6 leads to the proposal in [19], Section 7, for the shortfall risk contributions in case $n = 1$.

**Example 5.8** (Continuation of Example 3.3)
By Lemma 5.6 under Assumption (S) for $i = 1, \ldots, d$ the mappings $a_i^{(n)} : \mathbb{R}\backslash\{0\} \times \mathbb{R}^{d-1} \to \mathbb{R}$ with

(5.21) $$a_i^{(n)}(u) \overset{\text{def}}{=} \frac{\partial r^{(n)}}{\partial u_i}(u) = \frac{\partial T_{\alpha,n}}{\partial u_i}(u)$$

are well-defined. They provide a vector field of risk contributions $a^{(n)} = (a_1^{(n)}, \ldots, a_d^{(n)})$ which by Proposition 3.5 b) satisfies

$$n \, r^{(n)}(u) = \sum_{i=1}^{d} u_i \, a_i^{(n)}(u), \quad u \in U,$$

and is suitable for performance measurement with $r$ by Theorem 4.4. Moreover, the $a_i^{(n)}(u)$ can also be calculated via

(5.22) $$a_i^{(n)}(u) = n \, \mathrm{E}\left[ X_i \, Z(u)^{n-1} \mid Z(u) \geq Q_\alpha(u) \right]. \qquad \square$$

Observe that, if Assumption (S) does not hold, in the cases of quantile and shortfall based risk measures we can define risk contributions by (5.12) and (5.22) resp. The contributions defined in this way might also have a good chance to be suitable with their corresponding risk measures.

# 6 A mean-quantile CAPM

Consider assets $0, 1, \ldots, d$. Assets $1, \ldots, d$ are risky and earn margins $m_i > 0, i = 1, \ldots, d$, plus the risk-free interest rate. Asset 0 is risk-free and earns exactly the risk-free rate (i.e. $m_0 = 0$). We regard portfolios $(u_0, u_1, \ldots, u_d)$ which are weighted compositions of the assets $0, 1, \ldots, d$. The weight $u_0 \in \mathbb{R}$ of asset 0 is arbitrary, i.e. the investor may lend or borrow at the risk-free rate, whereas the weights $u_i$ of assets $1, \ldots, d$ are non-negative, i.e. short selling with these assets is not allowed.

The classical Markowitz portfolio theory (cf. [21]) tells us how to find optimal portfolio weights for a given risk level $R$. The risk in this theory is measured with the variance of the portfolio return. The optimal weight vector $(u_0, u_1, \ldots, u_d)$ maximizes the expected portfolio return $\sum_{i=1}^{d} m_i u_i$ exceeding the risk-free rate under the restrictions $r(u_0, u_1, \ldots, u_d) = R$ and $\sum_{i=0}^{d} u_i = 1$.

For the classical case, the Tobin separation theorem says that there are weights $v_i \in [0, 1], i = 1, \ldots, d$, with $\sum_{i=1}^{d} v_i = 1$ such that for each fixed risk value $R$ the optimal portfolio can be represented as $(1 - h, h\, v_1, \ldots, h\, v_d)$ with $h$ depending on $R$. The vector $(v_1, \ldots, v_d)$ can be determined by maximizing the *Sharpe ratio*, i.e. the quotient of the portfolio return minus risk-free rate and the standard deviation of the portfolio return.

In the sequel, we study the connection between separation and Sharpe ratio in the context of more general risk measures.

**Definition 6.1**

*Let $r$ be a function $\mathbb{R} \times [0, \infty)^d \to \mathbb{R}$. We say that a vector $(v_1, \ldots, v_d) \in [0, 1]^d$ with $\sum_{i=1}^{d} v_i = 1$ has the* two fund separation *property for $r$ if for each $C > 0$ and $R > 0$ there is an $h > 0$ such that*

$$(6.1) \qquad R \;=\; r(C - h, h\, v_1, \ldots, h\, v_d) \quad and$$

$$(6.2) \quad h \sum_{i=1}^{d} m_i v_i \;=\; \sup \left\{ \sum_{i=1}^{d} m_i u_i \;\middle|\; u_0 \in \mathbb{R}, u_i \in [0, \infty),\, i = 1, \ldots, d, \right.$$
$$\left. \sum_{i=0}^{d} u_i = C,\, r(u_0, u_1, \ldots, u_d) = R \right\}. \qquad \square$$

Property (6.1) means that an investor whatever be his initial wealth $C$, can reach exactly the risk level $R$ by apportioning his wealth on the risk-free asset and the portfolio $(v_1, \ldots, v_d)$ of assets $1, \ldots, d$ only. Note that asset 0 is risk-free and does not contribute to the excess return on the portfolio over the risk-free rate but may influence the portfolio risk. This could be regarded as a leverage effect. Examples for risk measures exhibiting this behaviour are the MLPM's (mean-lower partial moments) with arbitrary target rates which are studied in [14]. (6.2) is just the optimality from the separation theorem.

In general, characterization of the two fund separating vectors from Definition 6.1 seems difficult. However, in situations as that with the VaR risk measure from Example 3.2, a characterization analogous to that with the Sharpe ratio is possible. The proof of the following proposition is easy and therefore is omitted.

**Proposition 6.2**

*Let $r : \mathbb{R} \times [0, \infty)^d \to \mathbb{R}$ be given by $r(u_0, u_1, \ldots, u_d) = \rho(u_1, \ldots, u_d)$, where $\rho : [0, \infty)^d \to \mathbb{R}$ is $\tau$-homogeneous for some $\tau > 0$. Let $(v_1, \ldots, v_d) \in [0, \infty)^d$ be a vector with $\sum_{i=1}^{d} v_i = 1$ and $\rho(v_1, \ldots, v_d) > 0$.*

*Then $(v_1, \ldots, v_d)$ is two fund separating for $r$ if and only if*

$$(6.3) \quad \frac{(\sum_{i=1}^{d} m_i \, v_i)^\tau}{\rho(v_1, \ldots, v_d)} = \sup \left\{ \frac{(\sum_{i=1}^{d} m_i \, u_i)^\tau}{\rho(u_1, \ldots, u_d)} \ \Big| \ u_i \in [0, \infty), \ i = 1, \ldots, d, \right.$$

$$\left. \rho(u_1, \ldots, u_d) > 0 \right\}. \qquad \square$$

Differentiating the left-hand side of (6.3) we obtain a linear relationship between the asset margins and the margin of the two fund separating portfolio.

### Corollary 6.3

*Let $r, \rho$ and $\tau$ be as in Proposition 6.2. Let $(v_1, \ldots, v_d) \in (0, 1)^d$ with $r^\star \overset{\text{def}}{=} \rho(v_1, \ldots, v_d) > 0$ have the two fund separation property and assume that $\rho$ is differentiable in the point $(v_1, \ldots, v_d)$. Set $m^\star \overset{\text{def}}{=} \sum_{i=1}^{d} m_i \, v_i$.*

*Then we have*

$$(6.4) \qquad m_i = m^\star (\tau \, r^\star)^{-1} \frac{\partial \rho}{\partial u_i}(v_1, \ldots, v_d), \quad i = 1, \ldots, d. \qquad \square$$

### Remark 6.4

(i) If $\rho$ is continuous and positive on the compact set $\{(u_1, \ldots, u_d) \in [0, \infty)^d \mid \sum_{i=1}^{d} u_i = 1\}$ then Proposition 6.2 implies the existence of a two fund separating portfolio.

(ii) When $(v_1, \ldots, v_d)$ denotes a market portfolio in the sense of the CAPM, the number $m^\star$ is the excess return on the market over the risk-free interest rate. (6.4) then may be considered as generalized CAPM-equation (cf. (9.11) in [21]) since it expresses the excess return on asset $i$ over the risk-free rate as the excess return on the market times the *sensitivity* of asset $i$ to the market.

(iii) Given a statistical portfolio model as in Section 3 which satisfies Assumption (S) and appropriate moment conditions, corollary 6.3 implies Theorem 4 in [6] and mean-quantile CAPM-formulae.

(iv) When $\rho(v_1, \ldots, v_d) - (\sum_{i=1}^{d} m_i \, v_i)^\tau$ is positive, the two fund separating portfolio $(v_1, \ldots, v_d)$ can also be characterized by

$$(6.5) \ (v_1, \ldots, v_d) =$$

$$\arg \sup \left\{ \frac{\sum_{i=1}^{d} m_i \, u_i}{\sqrt[\tau]{\rho(u_1, \ldots, u_d)} - \sum_{i=1}^{d} m_i \, u_i} \ \Big| \ u_i \in [0, \infty), \ i = 1, \ldots, d, \right.$$

$$\left. \rho(u_1, \ldots, u_d) > (\sum_{i=1}^{d} m_i \, u_i)^\tau \right\}.$$

This gives the connection between the performance considerations from Section 4 and the Markowitz-like point of view we take in this section. Moreover, we see that when measuring performance in a risk-adjusted manner one should use a 1-homogeneous risk measure. $\qquad \square$

# 7 Concluding remark

This paper contains bad news and good news. The bad news are that some commonly recommended methods of allocating risk to subportfolios or business lines are suspicious of rendering misleading information (Examples 4.5 and 5.2). The good news say that there is a right way to do the allocation (Theorem 4.4). This way seems practicable even for more sophisticated risk measures as the VaR (Lemma 5.3 and (4.12)). Recent simulation results in [13] point out that direct estimation of the conditional mean in (5.12) seems feasible.

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