Abstract.

A method of evaluation of the multi-period economic risk capital of a life portfolio, defined as the value-at-risk or conditional value-at-risk measure associated to the multi-period aggregate surplus, is presented. It is applied to a portfolio of unit-linked endowment policies with guarantee. Using a simple but reliable gamma approximation of the insurance risk, it is shown that the guaranteed unit-linked endowment insurance performs better than the traditional endowment insurance on a risk-adjusted return scale. The risk-adjusted return on capital, abbreviated RAROC, that is the expected gain per unit of economic risk capital is greater for the guaranteed unit-linked contract than for the traditional one.

Key words: aggregate surplus, value-at-risk, conditional value-at-risk, RAROC, unit-linked insurance, endowment insurance

1. Introduction.

The present paper shows how the multi-period economic risk capital of a life portfolio can be evaluated. Defining the economic risk capital as the value-at-risk or conditional value-at-risk associated to the multi-period aggregate surplus of a life portfolio, the last quantity must be first determined. For this, the multi-period aggregate surplus is divided into three components, namely the investment risk, the insurance risk and the insurance margin, where each component is modeled separately. Our application concerns the evaluation of the economic risk capital of a portfolio of unit-linked endowment policies with guarantee and its value based comparison with a portfolio of traditional endowment policies. Using a simple but reliable gamma approximation of the insurance risk component, it is shown that the guaranteed unit-linked contract performs better than the traditional one on a risk-adjusted return scale. Indeed, the risk-adjusted return on capital, abbreviated RAROC, that is the expected gain per unit of economic risk capital is greater for the guaranteed unit-linked product than for the traditional one. A more detailed outline follows.

In Section 2 we consider a simple stochastic model of a life portfolio in a multi-period setting, which allows one to determine its accumulated aggregate surplus over a finite number of insurance periods as well as its associated economic risk capital. The aggregate surplus is decomposed in three summands. The first term represents the possible random investment gain or profit in excess of the guaranteed technical interest and its negative value is called investment risk. The second term represents the possible random insurance loss as deviation of the random accumulated value of the aggregate claims in each period from its expected value and is called insurance risk. Finally, the third term represents the expected accumulated insurance surplus induced by the risk premium margins and is called insurance margin. In
practical evaluations, it is necessary to specify a model of the accumulated aggregate claims and a distribution for the random accumulated return on investment.

Section 3 recalls the notion of economic risk capital (ERC) derived from the well established value-at-risk (VaR) and conditional value-at-risk (CVaR) capital requirements. We point out diversification effects between the insurance and investment risks, and show how the overall ERC can be allocated to the insurance and investment components using a simple covariance principle.

Section 4 surveys the analysis and pricing of unit-linked contracts with guarantees, whose main feature is the inclusion of random benefits as opposed to deterministic benefits in traditional life insurance. Inspired by the developments in Aase and Persson(1994), we offer an analytical treatment close to the traditional viewpoint, which can be easily applied to the evaluation of economic risk capital along the presented approach.

Section 5 illustrates the actuarial use of our results. In Section 5.1 we show by example that the distribution of the random multi-period aggregate loss of a life portfolio can be well approximated using a gamma distribution assumption for the insurance risk. Making this approximation, we compare in Section 5.2 the traditional endowment insurance with the guaranteed unit-linked endowment insurance and obtain an improved risk-adjusted performance for the latter.

2. The aggregate surplus of a life portfolio.

We consider a simple stochastic model of a life portfolio in a multi-period setting, which allows one to determine its accumulated aggregate surplus over a finite number of insurance periods. This quantity is essential for a life insurance company because it is the main ingredient required to evaluate the economic risk capital associated to life insurance portfolios.

Our simplified insurance business model takes into account the insurance risk associated to random insurance claims and the investment risk associated to random returns on invested capital. It is assumed throughout that the random claims and returns are stochastically independent. The management risk associated to administration expenses and the attitude to risk are not taken into account here. The technical values of a traditional life portfolio can be summarized as follows (e.g. Bowers et al.(1986), p. 209):

\[ \pi_t^R \quad : \quad \text{the risk premiums in period } [t-1,t] \text{ due at time } t-1 \]
\[ \pi_t^S \quad : \quad \text{the saving premiums in period } [t-1,t] \text{ due at time } t-1 \]
\[ \pi_t = \pi_t^R + \pi_t^S \quad : \quad \text{the net premiums} \]
\[ J \quad : \quad \text{the actuarial reserves required at time } t \]
\[ S_t \quad : \quad \text{the random aggregate claims in period } [t-1,t] \text{ due at time } t \]
\[ I_t \quad : \quad \text{the random return on investment in period } [t-1,t] \]
\[ i \quad : \quad \text{the one-period constant technical interest rate} \]
\[ r = 1 + i \quad : \quad \text{the one-period accumulated technical interest rate} \]
\[ v = r^{-1} \quad : \quad \text{the discount rate} \]

Recall some well-known relationships. The risk premiums are equal to the discounted value of the expected aggregate claims and the loading on aggregate claims (to fix ideas think of the loading as a multiple of the standard deviation of aggregate claims). In formulas one has

\[ \pi_t^R = v \cdot (E[S_t] + \ell[S_t]). \quad (2.1) \]
The discounted reserves at time \( t \) consists of the reserves at time \( t-1 \) and the saving premiums, that is
\[
v_t V = v_{t-1} V + \pi_t^S. \tag{2.2}
\]

One is interested in the aggregate surplus of the life portfolio at a future time \( T \), that is after a number \( T \) of insurance periods. The random value of the liability \( L_T \) at time \( T \) of the portfolio consists of the actuarial reserves at time \( T \) and the random accumulated value of the aggregate claims in each period \([t-1,t], \ t = 1, \ldots, T\), that is
\[
L_T = V + \sum_{t=1}^{T} r^{T-t} \cdot S_t. \tag{2.3}
\]

Discounted at the technical interest rate, the present value of the assets \( A_0 \) at time \( t = 0 \) of the life portfolio consists of the actuarial reserves and the discounted value of all future premiums, that is
\[
A_0 = V + \sum_{t=1}^{T} v^{t-1} \cdot \pi_t. \tag{2.4}
\]

The aggregate surplus at time \( T \), defined as difference between assets and liabilities, is denoted and equal to \( G_T = A_0 \cdot R_T - L_T \), where \( R_T = \prod_{t=1}^{T} (1 + I_t) \) represents the random accumulated return on investment over the period \([0, T]\). Using (2.2) one obtains through induction the actuarial reserves at time \( T \)
\[
rV = r^{T-0} V + \sum_{t=1}^{T} r^{T-t} \cdot \pi_t^S. \tag{2.5}
\]

Using (2.1)-(2.5) the aggregate surplus can be rewritten as
\[
G_T = A_0 \cdot (R_T - r^T) - \sum_{t=0}^{T-1} r^{T-t-1}(S_{t+1} - E[S_{t+1}]) + \sum_{t=0}^{T-1} r^{T-t-1} \ell[S_{t+1}]. \tag{2.6}
\]

The first term, abbreviated \( G_T^{\text{inv}} \), represents the possible random investment gain in excess of the guaranteed technical interest and its negative value is called investment risk. The second term represents the possible random insurance loss as deviation of the random accumulated value of the aggregate claims in each period from its expected value and is called insurance risk, abbreviated \( V_T^{\text{ins}} \). Finally, the third term represents the expected accumulated insurance surplus at time \( T \) provided by the risk premium margins and is called insurance margin, abbreviated \( M_T^{\text{ins}} \). For (2.6) one uses the following short hand notation
\[
G_T = G_T^{\text{inv}} - V_T^{\text{ins}} + M_T^{\text{ins}}. \tag{2.7}
\]

In practice, it is necessary to specify a model of the accumulated aggregate claims (e.g. Hürlimann(2001a) for an individual multi-period life model with one cause of decrement) and a distribution for the random accumulated return on investment (e.g. a normal, elliptical, log-normal distribution, etc.).
3. **Multi-period economic risk capital.**

One of the major problems faced by insurance companies is the determination of capital requirement associated to the insurance and investment risks of portfolios of insurance policies. The basic approach consists to apply an appropriate *risk measure* that takes into account the shape of the profit and loss distribution, especially its right tail. The well-established value-at-risk (VaR) capital requirement (quantile reserve defined as percentile of the loss distribution) may fail to be subadditive (and thus fail to stimulate diversification) and does not take into account the severity of an incurred adverse loss event. The encountered deficiencies are captured by the notion of *coherent risk measure* largely discussed in the recent literature (e.g. Arztnert et al. (1997/99), Arztnert(1999), Wirch (1999), Wirch and Hardy(1999), Delbaen(2000), Testuri and Uryasev(2000), Acerbi (2001), Acerbi and Tasche(2001a/b)). Besides VaR our coherent risk measure will be the *conditional value-at-risk* measure (CVaR).

The *economic risk capital* (ERC) associated to a loss random variable $X$ with distribution function $F_X(x)$ using these two approaches is defined as follows:

**Value-at-Risk (VaR)**

$$VaR_\alpha[X] = Q_\alpha(\alpha) = \inf \{ x \mid F_X(x) \geq \alpha \}$$

This value is the maximum possible loss, which is not exceeded with the probability $\alpha$ (in practice $\alpha = 95\%, 99\%$). VaR is in general not subadditive and not a coherent risk measure.

**Conditional Value-at-Risk (CVaR)**

$$CVaR_\alpha[X] = E[X \mid X > VaR_\alpha[X]]$$

The conditional expected loss given the loss exceeds its value-at-risk represents the “average of the $100\alpha\%$ worst losses” in a random sample of losses. One has the useful formula

$$CVaR_\alpha[X] = VaR_\alpha[X] + m_X[VaR_\alpha[X]] = VaR_\alpha[X] + \frac{1}{\varepsilon} \pi_X[VaR_\alpha[X]],$$

where $m_X(x) = E[X - x \mid X > x]$ is the mean excess function, $\pi_X(x) = S_X(x) \cdot m_X(x)$ is the stop-loss transform, $S_X(x) = 1 - F_X(x)$ is the survival function, and $\varepsilon = 1 - \alpha$ is interpreted as loss probability. Some properties of CVaR, which is a coherent risk measure, are discussed in several recent papers (e.g. Testuri and Uryasev(2000), Acerbi(2001), Acerbi and Tasche(2001a/b), Hürlimann(2001b), Kusuoka(2001), Rockafellar and Uryasev(2001)).

According to (2.7) the random aggregate loss of a life portfolio at time $T$ is equal to

$$V_T = V_T^{inv} + V_T^{ins} - M_T^{inv},$$  \hspace{1cm} (3.1)$$

where $V_T^{inv} = -G_T^{inv} = A_0 \cdot (r^T - R_T)$ represents the possible random investment loss, which is positive in case the technical interest cannot be guaranteed. To be able to cover a possible loss $V_T > 0$ with a high probability, an insurance company borrows at time $t = 0$ the amount $ERC_0[V_T]$, called *economic risk capital*. At time $T$, interest is due at the periodic rate $i_C$. To
guarantee with certainty the value of the borrowed capital at time \( T \), the amount \( ERC_0[V_T] \) is invested at the periodic riskless rate \( i_j < i_C \). The accumulated value of the economic risk capital at time \( T \) is herewith

\[
ERC_T[V_T] = ERC_0[V_T] \cdot (1 + r^T - r^C),
\]  (3.2)

where one sets \( r_j = 1 + i_j \), \( r_C = 1 + i_C \). Applying the above approaches two functionals must be evaluated:

\[
ERC^\alpha_T[V_T] = VaR_\alpha[V_T] \quad \text{(value-at-risk approach),} \\
ERC^\alpha_T[V_T] = CVaR_\alpha[V_T] \quad \text{(conditional value-at-risk approach).} 
\]  (3.3)  (3.4)

To examine and point out diversification effects between the insurance and investment risks, it is of great importance to look separately at the ERC of the insurance loss, called insurance economic risk capital and abbreviated I-ERC, and at the ERC of the investment loss, called market economic risk capital and abbreviated M-ERC. Similarly to (3.3) and (3.4) one has herewith the alternative functionals

\[
I - ERC^\alpha_T[V^\text{ins}_T] = VaR_\alpha[V^\text{ins}_T], \quad M - ERC^\alpha_T[V^\text{inv}_T] = VaR_\alpha[V^\text{inv}_T], \\
I - ERC^\alpha_T[V^\text{ins}_T] = CVaR_\alpha[V^\text{ins}_T], \quad M - ERC^\alpha_T[V^\text{inv}_T] = CVaR_\alpha[V^\text{inv}_T].
\]  (3.5)  (3.6)

In general, one postulates the subadditive property

\[
ERC^\alpha_T[V^\text{ins}_T + V^\text{inv}_T] \leq I - ERC^\alpha_T[V^\text{ins}_T] + M - ERC^\alpha_T[V^\text{inv}_T].
\]  (3.7)

The non-negative difference in (3.7) represents a diversification effect in form of an excess risk capital, which is no more required in case of separate evaluations of I-ERC and M-ERC. There exists several risk allocation principles, which apportion the non-negative diversification effect to the insurance and investment risk components (e.g. Tasche(2000), Delbaen and Denault(2000), Hürlimann(2001c)). According to a simple covariance principle, the allocated risk contributions \( I-ERC^* \) and \( M-ERC^* \) such that

\[
ERC^\alpha_T[V_T] = I - ERC^\alpha_T[V^\text{ins}_T] + M - ERC^\alpha_T[V^\text{inv}_T] = I - ERC^\alpha_T[V^\text{ins}_T] + M - ERC^\alpha_T[V^\text{inv}_T] - M^\text{ins}_T
\]  (3.8)

are determined by the formulas

\[
I - ERC^\alpha_T[V^\text{ins}_T] = E[V^\text{ins}_T] + \frac{Cov[V^\text{ins}_T, V_T]}{Var[V_T]} \left( ERC^\alpha_T[V_T] - E[V_T] \right),
\]  (3.9)

\[
M - ERC^\alpha_T[V^\text{inv}_T] = E[V^\text{inv}_T] + \frac{Cov[V^\text{inv}_T, V_T]}{Var[V_T]} \left( ERC^\alpha_T[V_T] - E[V_T] \right).
\]  (3.10)

The interested reader should note that (3.9)-(3.10) can be interpreted as a CAPM like principle. This general principle has been first advocated in Borch(1982/90) and can be generalized applying at least four different approaches, as shown in Hürlimann(1998).
4. **Analysis and pricing of unit-linked life insurance with guarantee.**

The analysis and pricing of unit-linked contracts with guarantees, whose main feature is the inclusion of random benefits as opposed to deterministic benefits in traditional life insurance, has been discussed in the literature since about 25 years by many researchers, in particular Brennan and Schwartz(1976/79a/b), Boyle and Schwartz(1977), Corby(1977), Delbaen(1986), Delvaux and Magnée(1991), Bacinello and Ortu(1993), Aase and Persson(1994), Nielsen and Sandmann(1995), Kurz(1996).

Inspired by the developments in Aase and Persson(1994), we offer an analytical treatment of unit-linked contracts with guarantee, which is very close to the traditional viewpoint, and thus can be easily applied to the evaluation of economic risk capital following the approach in Sections 2 and 3. Our attention focuses on combinations of pure endowment and term insurance contracts, especially unit-linked endowment contracts with guarantee against constant periodic premiums. For clearness the presentation is divided into four Subsections.

4.1. **Unit-linked contracts.**

The description is based on the following quantities:

- $x$: age at entry of a policy-holder
- $s$: age at expiration of a contract
- $T = s - x$: term of a contract
- $t$: current time of valuation
- $G(t)$: guaranteed death benefit at time $t < T$
- $G(T)$: guaranteed benefit at expiration of a contract
- $v$: technical discount rate
- $S(t)$: random market value at time $t$ of one share of the unit-linked reference portfolio
- $N(t)$: prescribed number of shares at time $t$ of the reference portfolio included in the benefit
- $j$: expected return of the reference portfolio over the time horizon $[0,T]$
- $\sigma$: volatility of the reference portfolio

One observes that the guaranteed benefit at time $t$ is covered through shares of the reference portfolio provided one has

$$ N(t)S(t) = G(t). \quad (4.1) $$

However, at time $t = 0$ of underwriting the future value $S(t)$ of the reference portfolio is unknown. Therefore, for the purpose of pricing, we assume that the expected value of the shares in the reference portfolio covers the guaranteed benefit, that is $N(t)E[S(t)] = G(t)$. It follows that the prescribed number of shares is determined by

$$ N(t) = G(t)w', \quad w = (1 + j)^{-1}. \quad (4.2) $$

If the market value of the shares in the reference portfolio exceeds the guaranteed benefit at death or expiration, then the difference is paid as bonus to the beneficiary of the contract. The financial payoff at time $t$ satisfies the relationship
max\{N(t)S(t), G(t)\} = G(t) + (N(t)S(t) - G(t))_+ . \quad (4.3)

With (4.3) the payoff is decomposed in a deterministic payment $G(t)$ and a stochastic payment from a call-option on the market value of the shares with exercise price $G(t)$.

### 4.2. Pricing of the unit-linked endowment contract with guarantee.

The benefit decomposition (4.3) suggests to represent technical values of unit-linked contracts as sums of technical values of traditional contracts with deterministic payoff $G(t)$ and of technical values of contracts with stochastic call-option payoff $(N(t)S(t) - G(t))_+$. The endowment insurance is the superposition of a term insurance and a pure endowment insurance. Therefore, each technical value can be additionally decomposed in a term and pure endowment component. Following this procedure, the two-fold decomposition of the relevant technical values with their notations are summarized in the Tables 4.1 and 4.2.

**Table 4.1**: Decomposition of single premiums

<table>
<thead>
<tr>
<th>benefit</th>
<th>insurance</th>
<th>term insurance</th>
<th>pure endowment</th>
<th>endowment</th>
</tr>
</thead>
<tbody>
<tr>
<td>traditional</td>
<td>$TI_x$</td>
<td>$TPE_x$</td>
<td>$TE_x$</td>
<td></td>
</tr>
<tr>
<td>unit-linked call-option</td>
<td>$TIC_x$</td>
<td>$TPEC_x$</td>
<td>$TEC_x$</td>
<td></td>
</tr>
<tr>
<td>guaranteed unit-linked</td>
<td>$TIG_x$</td>
<td>$TPEG_x$</td>
<td>$TEG_x$</td>
<td></td>
</tr>
</tbody>
</table>

**Table 4.2**: Decomposition of actuarial reserves

<table>
<thead>
<tr>
<th>benefit</th>
<th>insurance</th>
<th>term insurance</th>
<th>pure endowment</th>
<th>endowment</th>
</tr>
</thead>
<tbody>
<tr>
<td>traditional</td>
<td>$V_x^T$</td>
<td>$V_x^{PE}$</td>
<td>$V_x^E$</td>
<td></td>
</tr>
<tr>
<td>unit-linked call-option</td>
<td>$V_x^{TIC}$</td>
<td>$V_x^{PEC}$</td>
<td>$V_x^{EC}$</td>
<td></td>
</tr>
<tr>
<td>guaranteed unit-linked</td>
<td>$V_x^{TIG}$</td>
<td>$V_x^{PEG}$</td>
<td>$V_x^{EG}$</td>
<td></td>
</tr>
</tbody>
</table>

The remaining required technical values for each component are derived from the single premium, written in general notation as $E_x$, the actuarial reserves $V_x$, $V_{x+t}$, and the discount rate $\nu$ using the following formulas:

\[
\pi_x = \frac{E_x}{\bar{a}_{\nu,T}} \quad : \text{periodic net premium} \quad (4.4)
\]

\[
\pi_x^s = \nu \cdot V_{x+t} - V_x \quad : \text{saving premium in period } [t-1,t] \quad (4.5)
\]

\[
\pi_x^R = \pi_x - \pi_x^s \quad : \text{risk premium in period } [t-1,t] \quad (4.6)
\]

\[
S_x = \pi_x^R \cdot (\nu q_{x+t})^{-1} \quad : \text{sum at risk in period } [t-1,t] \quad (4.7)
\]
4.3. Analytical formulas.

The valuation of the traditional insurance contracts with deterministic payoff is classical and well-known. Present values of the unit-linked call-option contracts are determined using Black-Scholes formula, where the riskless rate is set equal to the technical interest rate. It follows that the value of the call-option at time $t$ with payoff $(N(u)S(u) - G(u))$, at time $u > t$, abbreviated $C(t, u)$, is given by (use the relationship (4.2)):

$$C(t, u) = N(u)S(t)\Phi\left[d_1(t, u)\right] - G(u)\nu^{u-t}\Phi\left[d_2(t, u)\right]$$

(4.8)

$$d_1(t, u) = \frac{\ln\left(\frac{N(u)S(t)}{G(u)}\right) + \left(\delta + \frac{1}{2}\sigma^2\right)(u-t)}{\sigma\sqrt{u-t}}$$

$$d_2(t, u) = d_1(t, u) - \sigma\sqrt{u-t}, \quad \delta = -\ln(\nu).$$

At the time $t = 0$ of underwriting, the future value $S(t)$ is unknown. As in Section 4.1 one assumes in formula (4.8) that the future value at time $t$ of a share of the reference portfolio coincides with its expected value, that is $S(t) = (1+j)^T$. Under this assumption the required technical values are calculated explicitly as follows:

**Single premiums**

$$TI_{x:T} = \sum_{k=0}^{T-1} G(k+1)x_p \cdot q_{x+k}$$

(4.9)

$$\tau PE_x = \nu^T \cdot G(T)$$

(4.10)

$$TIC_{x:T} = \sum_{k=0}^{T-1} C(0,k+1)x_p \cdot q_{x+k}$$

(4.11)

$$\tau PEC_x = \nu^T \cdot C(0,T)$$

(4.12)

$$\tau E_x = TI_{x:T} + \tau PE_x$$

(4.13)

$$\tau EC_x = TIC_{x:T} + \tau PEC_x$$

(4.14)

$$TIG_{x:T} = TI_{x:T} + TIC_{x:T}$$

(4.15)

$$\tau PEG_x = \nu^T \cdot PEC_x$$

(4.16)

$$\tau EG_x = \tau E_x + \tau EC_x = TIG_{x:T} + \tau PEG_x$$

(4.17)

**Actuarial reserves**

$$V^T_x = \sum_{k=0}^{T-1} G(k+1) \cdot \nu^{k+1-t} \cdot p_{x+k} \cdot q_{x+k} - \sum_{k=0}^{T-1} \pi^T_{x} \cdot \nu^{k+1-t} \cdot p_{x+k},$$

(4.18)

$t = 0, \ldots, T - 1, \quad V^T_x = 0$

$$V^{PE}_x = -p_{x+t} \cdot \nu^{t} \cdot G(T) - \sum_{k=0}^{T-1} \pi^T_{x} \cdot \nu^{k+1-t} \cdot p_{x+k},$$

(4.19)

$t = 0, \ldots, T - 1, \quad V^{PE}_x = G(T)$
\[ V_x^{TIC} = \sum_{k=0}^{T-1} C(t,k+1) \cdot q_{x+k} \cdot P_{x+t} - \sum_{k=0}^{T-1} \pi_x^{TIC} \cdot v^{k-1} \cdot q_{x+k} \cdot P_{x+t}, \quad t = 0, \ldots, T-1, \quad T V_x^{TIC} = 0 \]

\[ V_x^{PEC} = \sum_{k=0}^{T-1} \pi_x^{PEC} \cdot v^{k-1} \cdot q_{x+k} \cdot P_{x+t}, \quad t = 0, \ldots, T-1, \]

\[ V_x^{PEC} = E[(N(T)S(T) - G(T))] = G(T) \cdot \Phi\left(\frac{T}{\sigma}\right) - 1 \]

\[ V_x^E = V_x^{TIC} + V_x^{PEC} \]

\[ V_x^{EC} = V_x^{TIC} + V_x^{PEC} \]

\[ V_x^{EC} = V_x^{TIC} + V_x^{PEC} \]

\[ V_x^{EG} = V_x^{TIC} + V_x^{PEC} \]

\[ V_x^{EG} = V_x^{TIC} + V_x^{PEC} \]

### 4.4. A numerical illustration.

To illustrate numerically the obtained formulas, consider a unit-linked endowment policy with guaranteed payment \( G(t) = 100, t = 1, \ldots, T \), for a man aged \( x = 50 \) at entry with a term of \( T = 10 \) years. To calculate single premiums and actuarial reserves we use second order probabilities of death (here 60% of the Swiss Life Table GKM80) and a technical interest rate \( i = 2.5\% \). The expected return and the volatility of the reference portfolio are assumed to take the values \( j = 7\% \) and \( \sigma = 15\% \). Numerical results for the various quantities of interest are summarized in the Tables below.

**Table 4.3**: Single premiums

<table>
<thead>
<tr>
<th>product</th>
<th>Term insurance</th>
<th>Pure endowment</th>
<th>Endowment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Traditional</td>
<td>7.879</td>
<td>70.960</td>
<td>78.839</td>
</tr>
<tr>
<td>Unit-linked call</td>
<td>0.342</td>
<td>2.652</td>
<td>2.994</td>
</tr>
<tr>
<td>Unit-linked guarantee</td>
<td>8.221</td>
<td>73.612</td>
<td>81.833</td>
</tr>
</tbody>
</table>

**Table 4.4**: Annual net premiums

<table>
<thead>
<tr>
<th>product</th>
<th>Term insurance</th>
<th>Pure endowment</th>
<th>Endowment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Traditional</td>
<td>0.908</td>
<td>8.179</td>
<td>9.087</td>
</tr>
<tr>
<td>Unit-linked call</td>
<td>0.039</td>
<td>0.306</td>
<td>0.345</td>
</tr>
<tr>
<td>Unit-linked guarantee</td>
<td>0.947</td>
<td>8.485</td>
<td>9.432</td>
</tr>
</tbody>
</table>

**Table 4.5**: Actuarial reserves

<table>
<thead>
<tr>
<th>( t )</th>
<th>( V_x^{TI} )</th>
<th>( V_x^{PEC} )</th>
<th>( V_x^E )</th>
<th>( V_x^{TIC} )</th>
<th>( V_x^{PEC} )</th>
<th>( V_x^{EC} )</th>
<th>( V_x^{EG} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.341</td>
<td>8.433</td>
<td>8.774</td>
<td>0.022</td>
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Table 4.6: Risk premiums

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<th>$\pi_{R, PE}^{T_x}$</th>
<th>$\pi_{R, E}^{T_x}$</th>
<th>$\pi_{R, TIC}^{T_x}$</th>
<th>$\pi_{R, PEC}^{T_x}$</th>
<th>$\pi_{R, EC}^{T_x}$</th>
<th>$\pi_{R, EGR}^{T_x}$</th>
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</thead>
<tbody>
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</tr>
<tr>
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</tr>
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Table 4.7: Saving premiums

<table>
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<th>$t$</th>
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<th>$\pi_{S, PE}^{T_x}$</th>
<th>$\pi_{S, E}^{T_x}$</th>
<th>$\pi_{S, TIC}^{T_x}$</th>
<th>$\pi_{S, PEC}^{T_x}$</th>
<th>$\pi_{S, EC}^{T_x}$</th>
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<td>0.457</td>
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<td>0.161</td>
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<td>8.942</td>
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Table 4.8: Sums at risk

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<th>$t$</th>
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<th>$S_{PE}^{T_x}$</th>
<th>$S_{E}^{T_x}$</th>
<th>$S_{TIC}^{T_x}$</th>
<th>$S_{PEC}^{T_x}$</th>
<th>$S_{EC}^{T_x}$</th>
<th>$S_{EGR}^{T_x}$</th>
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</thead>
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<tr>
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<tr>
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<td>63.511</td>
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<td>4.475</td>
<td>−17.571</td>
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<td>40.659</td>
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<td>4.696</td>
<td>−12.237</td>
<td>−7.541</td>
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<td>43.864</td>
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<td>60.048</td>
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<tr>
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<td>−100</td>
<td>0</td>
<td>3.990</td>
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<td>−1033.44</td>
<td>−1033.44</td>
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</table>
Table 4.9: Number of shares and expected call-option payment

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<th>E[(N(t)S(t)−G(t)),]</th>
</tr>
</thead>
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<td>0</td>
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<td>93.458</td>
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<tr>
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<td>76.290</td>
<td>11.924</td>
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<tr>
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<tr>
<td>6</td>
<td>66.634</td>
<td>14.576</td>
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<td>58.201</td>
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<td>50.835</td>
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5. Traditional versus guaranteed unit-linked endowment insurance.

The purpose of the present Section is two-fold. First, in Section 5.1 we show by example how the distribution of the random multi-period aggregate loss of a life portfolio and the associated VaR and CVaR measures can be exactly evaluated and well approximated by the quantities obtained from a gamma distribution assumption for the insurance risk. Then, Section 5.2 is devoted to a comparison of the traditional endowment insurance with the guaranteed unit-linked endowment insurance. Within a value based management context, it is shown that the guaranteed unit-linked contract performs better than the traditional one on a risk-adjusted return scale. Indeed, the expected gain per unit of standard deviation, VaR or CVaR of a portfolio of identical policies is greater for the guaranteed unit-linked product than for the traditional one.

5.1. Gamma approximation of the insurance risk.

To determine the VaR and CVaR measures of a life portfolio, it is necessary to evaluate the distribution of the aggregate loss in (3.1). We follow the method applied in Hürlimann(2001a). The random accumulated return \( R_t \) on investment is modeled by a lognormal distribution with parameters \( \left[ \ln(1+j)−\frac{1}{2}\sigma^2 \right]T \) and \( \sigma\sqrt{T} \), with \( j \) and \( \sigma \) as in Section 4.1. The random accumulated aggregate claims \( S = \sum_{t=1}^{T} S_t \) follows a multi-period individual life model, whose distribution can be evaluated using the two stage recursive formulas of De Prill(1989) (see Hürlimann(2001a) for details). As a simple but reliable approximation of \( S \), one first replaces \( S_1,\ldots,S_T \) by independent copies \( S_{1}^\perp,\ldots,S_{T}^\perp \), and thus replace \( S \) by \( S^\perp = \sum_{t=1}^{T} S_{t}^\perp \). By Hürlimann(2001a), Section 4, this implies the stop-loss order relation \( S \leq_{st} S^\perp \), which means that \( S \) has been replaced by the slightly more dangerous \( S^\perp \) for which one has in particular \( E[S] = E[S^\perp] \) and \( Var[S] \leq Var[S^\perp] \). Then one approximates \( S^\perp \) by a gamma distributed random variable with mean \( E[S^\perp] \) and variance \( Var[S^\perp] \). As the following illustration shows, the obtained very tractable approximation yields enough accurate upper bounds for VaR and CVaR calculations.
Consider a portfolio of $N = 100$ traditional endowment policies with the characteristics of Section 4.4. The parameters of the gamma approximation are determined by the formulas

$$E[S^+] = N \sum_{k=1}^{T} \left( r^{T-k} \cdot k \cdot S^E_{x+k} \right) p_{x+k}q_{x+k-1}, \quad (5.1)$$

$$Var[S^+] = N \sum_{k=1}^{T} \left( r^{T-k} \cdot k \cdot S^E_{x+k} \right)^2 p_{x+k}q_{x+k-1} (1-p_{x+k}q_{x+k-1}), \quad (5.2)$$

where second order probabilities of death are used as in Section 4.4, and the sum at risk $k \cdot S^E_{x+k}$ is found in Table 4.8. By definition of the insurance margin in Section 2, one has $M^\text{ins}_T = \frac{2}{3} E[S^+]$ (recall that one assumes that 60% of the probabilities in the tariff suffice to cover the claims). Comparisons of exact and approximated values of VaR and CVaR for the confidence level $\alpha = 0.99$ are summarized in the Table 5.1.

<table>
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<tr>
<th>model</th>
<th>VaR</th>
<th>CVaR</th>
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</thead>
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<tr>
<td>exact distribution</td>
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<td>5749.48</td>
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<tr>
<td>gamma approximation</td>
<td>5411.53</td>
<td>6032.15</td>
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</table>

### 5.2. Risk-adjusted performance measurement.

The use of capital requirements using ERC risk measures like VaR and CVaR is of great importance in risk-adjustment performance measurement and has implications for the value based management of insurance products. Consider the random return per unit of ERC to a fixed confidence level $\alpha$ of a life portfolio at time $T$, called ERC gain ratio, which is defined by

$$GR^\alpha_T = \frac{G_T}{ERC^\alpha_T}$$

The expected value of the ERC gain ratio measures the risk-adjusted return on capital. This way of computing the return is commonly called RAROC (e.g. Matten(1996), p.59), and is here defined by

$$RAROC[GR^\alpha_T] = E[GR^\alpha_T]. \quad (5.4)$$

A related recent discussion of RAROC is found in Hürlimann(2001d). The use of RAROC as a value based management tool is straightforward. Indeed, if a product manager has to decide upon the more profitable of two life portfolios with accumulated aggregate surplus $G^1_T$ and $G^2_T$, a decision in favor of the first portfolio is taken if and only if one has $RAROC[GR^\alpha_T] \geq RAROC[GR^\alpha_T]$ at given confidence levels $\alpha$. This preference criterion tells us that a portfolio is preferred to another if its expected gain per unit of economic risk capital is greater. As an illustration we compare portfolios of $N$ identical policies with traditional endowment and guaranteed unit-linked endowment contracts under the assumptions of Section 4.4. We use a gamma approximation for the insurance risk as in Section 5.1 and note that $-S$ has to be assumed gamma distributed when $E[S] < 0$ as for our guaranteed unit-
linked policy. Our results for the confidence level $\alpha = 0.99$ by varying portfolio size $N$ are summarized in the Tables 5.2 and 5.3. It is remarkable that the guaranteed unit-linked contract performs better than the traditional contract on the RAROC scale. One observes that the same property holds if the economic risk capital measure is replaced by the simpler standard deviation risk measure.

**Table 5.2**: Measurement of traditional and guaranteed unit-linked endowment life insurance

<table>
<thead>
<tr>
<th>$N$</th>
<th>$\mu_G$</th>
<th>$\sigma_G$</th>
<th>VaR</th>
<th>CVaR</th>
<th>$\mu_G$</th>
<th>$\sigma_G$</th>
<th>VaR</th>
<th>CVaR</th>
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<td>6441</td>
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<td>5621</td>
<td>6420</td>
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</table>

**Table 5.3**: Comparison of risk-adjusted performance measures

<table>
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<th>$\frac{\mu_G}{\sigma_G}$</th>
<th>$\frac{\mu_G}{\text{VaR}}$</th>
<th>$\frac{\mu_G}{\text{CVaR}}$</th>
<th>$\frac{\mu_G}{\sigma_G}$</th>
<th>$\frac{\mu_G}{\text{VaR}}$</th>
<th>$\frac{\mu_G}{\text{CVaR}}$</th>
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<tbody>
<tr>
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<td>0.731</td>
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<td>0.976</td>
<td>0.763</td>
<td>1.146</td>
<td>1.003</td>
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<td>1.055</td>
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<td>0.971</td>
<td>0.770</td>
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**References.**


