Copula-Dependent Default Risk in Intensity Models

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Copula-Dependent Default Risk in Intensity Models

In this paper we present a new approach to incorporate dynamic default dependency in intensity-based default risk models. The model uses an arbitrary default dependency structure which is specified by the Copula of the times of default, this is combined with individual intensity-based models for the defaults of the obligors without loss of the calibration of the individual default-intensity models. The dynamics of the survival probabilities and credit spreads of individual obligors are derived and it is shown that in situations with positive dependence, the default of one obligor causes the credit spreads of the other obligors to jump upwards, as it is experienced empirically in situations with credit contagion. For the Clayton copula these jumps are proportional to the pre-default intensity. If information about other obligors is excluded, the model reduces to a standard intensity model for a single obligor, thus greatly facilitating its calibration. To illustrate the results they are also presented for Archimedean copulae in general, and Gumbel and Clayton copulae in particular. Furthermore it is shown how the default correlation can be calibrated to a Gaussian dependency structure of CreditMetrics-type.

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1. Introduction

Recently there has been strong interest in the development of accurate models for default dependencies. This interest is part of a general development towards an active and market-based management of credit risks in banks and other financial institutions. It is recognized that – unless the exposure is exceptionally large – individual defaults do not significantly affect the risk of the portfolio as long as they can be diversified. If on the other hand there are strong, systematic dependencies, even a portfolio containing a large number of small loans can be highly risky.

In the drive to securitize and trade loan and bond portfolios, several new derivative securities have been developed whose payoffs depend on the overall default behaviour of a whole portfolio of underlying loans or bonds. Prominent examples are basket credit derivatives and collateralised debt obligations (CDOs, CLOs, CBOs). These credit derivatives are actively traded which requires a methodology to measure default- and market risks on a day-by-day basis. A consistent model for default dependencies is essential to price and hedge these structures.

Furthermore, with the increasing volume in credit derivatives transactions, counterparty risk is recognized as a major source of risk which can affect the value of the credit protection bought. Here, too, a model for default dependencies is indispensable to assess and manage these risks, and again this must be a dynamic model.

In this paper we provide a modelling framework for default dependency which differs in several respects from other models in the literature. First, the model is a continuous-time *dynamic model*, and defaults and default probabilities evolve consistently within the model. In particular, the default probability of all obligors is continuously updated according to the observed default / no default behaviour of the other obligors. Secondly, the default dependency in the model can be specified in form of a copula of joint defaults at a given time-horizon, i.e. essentially static information. Third, the model can be directly calibrated to individual term structures of default intensities. Fourth, we allow for a much more general specification of the dependency between default events than other models in the literature. Furthermore, the analysis in this paper provides valuable insights in the connection between default dependencies and the joint dynamics of default intensities which are implicitly specified by specifying the default dependency. This information is particularly valuable in risk management.

Most of the literature on default correlation is concerned with the credit risk of large portfolios and based upon models like JPMorgan's Credit Metrics (1997) or Credit Suisse's Credit Risk+ (1997), see Crouhy et.al. (2000) for a comparison. The Credit Metrics approach has been refined in the factor-based default risk models by Vasicek (1987), Belkin et.al. (1998), Finger (1999) and Lucas et.al. (1999). These models also belong to the portfolio credit risk models, but some analytical tractability is regained by assuming a certain factor-dependence in the default mechanism. Here, the focus is the credit risk modelling for portfolio-management and regulatory issues. Therefore, simplifying assumptions are made on the individual default risk of the obligors, the dynamics of credit spreads and the term structures of credit spreads. Default correlations are modelled more carefully, usually with a copula based upon a multivariate normal distribution. Because of the broad focus on risk assessment, the models are generally not accurate and flexible enough to be calibrated to the market and to be used for the pricing of traded structures. Nevertheless, the data and the statistical analysis of default correlations on which these models are based is useful.

A weak point of all portfolio-based approaches is that they do not adequately resolve the *dynamics* of the value of the portfolio. For example, Credit Metrics is a one-period model with two points in time and a very coarse attempt to capture some market risk through rating changes. Credit Risk+ is set up in continuous-time but is only concerned with default/survival (and not price changes) in the portfolio. So far, there has been no model in the portfolio-based literature, which was able to consistently model default dependency and market price risk in one continuous-time modelling framework, although the drive towards securtisation and active management of credit risks as well as the rise in trading of basket credit derivatives and CDOs requires precise such a model. One contribution of this paper is to show how a *consistent* dynamic continuous-time model of defaults, default dependency and changes in default risk can be constructed for any given form of default dependency.

In order to point out the difficulties that may arise (and which have been largely ignored so far), consider two obligors A and B whose defaults are highly correlated. We do not know if or when they will default, but we have a joint distribution function $F(t_A, t_B)$ of the times of default τ_A of A and τ_B of B. The market prices of bonds issued by A will reflect the default probability until maturity of the bond of A, and the market prices of bonds issued by B the default probability of B. If obligor A defaults at some time t, a fundamental change occurs in our probability assessment of the default likelihood of B: While for $t < \tau_A$, the default probability of B was the default probability until maturity given that A will default *some time after* t, at τ_A it is the default probability given that A defaults *exactly now*. As A and B are highly correlated, the second conditioning information is much worse news on B's chances of survival, and this discrete change in information structure will cause a jump in the default probability of B at the time of default of A, and vice versa. While there have been some attempts to frame Credit Metrics into a continuous-time setup, these rational updating effects have been ignored so far. In this paper we give a precise description at what times under which conditions the presence of default dependence influences the dynamics of default intensities (and thus of default probabilities).

In contrast to the portfolio models which can capture individual credit risk only very coarsely, intensity-based single-name credit risk models provide a more flexible framework to model the dynamics and the term structure of credit risk. Usually they are based upon market variables such as credit spreads.

Most intensity models can be fitted easily to term structures of credit spreads and they have effectively become market standard in the pricing of standard credit derivatives such as credit default swaps. Important papers of this model class are Jarrow and Turnbull (1995) Duffie and Singleton (1999), Madan and Unal (1998), Lando (1998) and Schönbucher (1999; 1998).

Two approaches have been followed to extend these intensity-based models to incorporate default correlation and multiple defaults. The first, and simplest approach is to introduce correlation in the dynamics of the default intensities of the obligors, but to keep the models unchanged otherwise. This approach suffers from several disadvantages: The default correlations that can be reached with this approach are typically too low when compared with empirical default correlations, and furthermore it is very hard to derive and analyze the resulting default dependency structure. The problem of low default correlation becomes less severe when a large portfolio of different obligors is considered. Therefore these models are used either enterprise-wide (e.g. CreditRisk+) or for large CDOs (e.g. Duffie and Garleanu (1999)).

A refinement of this approach with more realistic default correlations is the infectious defaults model by Davis and Lo (1999a; 1999b; 2000), further developed by Jarrow and Yu (2000). There default intensities jointly jump upwards by a discrete amount at the onset of a credit crisis. While intuitively very appealing, deriving the survival probability for a single obligor is already a major task in this model which makes calibration very difficult. Furthermore, the estimation of the jump factor of the default intensities is also nontrivial, because it is not clear how this model can be calibrated to historical observations of default frequencies over a given time horizon. The advantage of this model over the Duffie/Singleton approach described below is, that here a realistic distribution of default times *over time* is achieved.

The second approach was introduced by Duffie and Singleton (1998b) and further developed by Kijima (2000; 2000). To reach stronger default dependencies, separate point processes are introduced that trigger *joint* default events at which several obligors default at the same time. For each possible joint default event, an intensity must be specified and calibrated. This approach achieves the goal of stronger default correlations but it suffers from the necessity to specify intensities for each possible joint default event (the number of these events grows exponentially with the number of obligors). Furthermore, it is unrealistic to assume that several obligors default *at the same time*. In particular for the pricing of CDOs a realistic time-structure of defaults is very important[1]. Credit contagion is also ruled out in this model: If an obligor is not hit by a default event, then its spreads will not change either.

The calibration of the model to individual term structures of credit spreads is also not trivial because the default of an individual obligor is driven by the first of a number of joint default events that all affect that obligor, so effectively a sum of interlinked intensities must be calibrated to an individual term structure of credit spreads.

Hull and White (2001) suggest to build a firm's value based model of default correlations, where defaults are triggered by time-dependent barriers in the firm's values, and default dependency is introduced by making the firm's value processes correlated. This model has several disadvantages: First, it is not clear if the model really reproduces a Gaussian dependency structure for the default times until a given time horizon (the barriers affect the joint distribution in a nonlinear way). Secondly, the proposed calibration mechanism will be numerically expensive and unstable (in order to reach nonzero initial credit spreads the barrier will have to have a infinitely negative slope at the initial date). Third, the numerical implementation will be slow as the full paths for all firm's value processes will have to be simulated, and finally, the model is restricted to a Gaussian dependency structure.

Copula functions have also been used to model default dependencies in the papers by Li (2000) and Frey and McNeil (2001). Li's paper is closest to this one, he models

[1] Many CDOs contain a cash reserve to protect the senior tranches of the transaction. This reserve is slowly built up from the cash flows of the underlying bonds or loans. If several defaults happen spread out over time this reserve may have been re-filled in the meantime, but if the defaults happen simultaneously the senior tranches will suffer losses.

the copula of the times-to-default of the different obligors, but without considering the dynamics of the default intensities (or in fact the default triggering mechanism). Frey and McNeil analyse in a fixed time-horizon setting the effects of the choice of different default-dependence copulas on the resulting returns distribution of a loan portfolio, which can be very important.

In this paper we propose a different way of extending the intensity-based approach to incorporate default correlations that preserves the advantages of both models described above: At the level of an individual obligor the model can be calibrated and intensity-dynamics can be specified like in any classical one-obligor intensity model. This is achieved without affecting the calibration of the other obligors or the dependency structure. The dependency structure of defaults is specified in form of a copula function which is completely independent from the dynamics of the individual default intensities, it can even be taken from a portfolio credit risk model of the firm's value type. Because the dependency structure of any vector of random variables is completely described by its copula function, and we have complete freedom in the specification of the copula that is used in the model, we can reproduce every possible dependency structure between the times of default of the obligors. Finally, the model still has a realistic time-structure of default times, because defaults do not happen at exactly the same time.

The rest of the paper is organized as follows:

In the next section we introduce some notation and review some basic facts on copula functions and default intensities that we will need later on.
This is followed by a description of the model setup and its analysis if only a single obligor is considered. It is shown that the model reduces to a classical intensity-model in this case.
Dependency between the different defaults is introduced in the following section. We analyse the resulting survival probabilities, default intensities and their dynamics, with a particular focus on the discrete changes in the survival probabilities and default intensities at the time of an obligor's default.
To demonstrate the modelling approach we then analyze some specific default dependency structures, some of which reduce the modelling effort significantly. We consider the class of Archimedean copula functions in general and two of its members: the Gumbel and the Clayton Copula. Furthermore, we consider the Gaussian copulas which arise from portfolio credit risk factor models like CreditMetrics or Vasicek (1987).
The following section shows briefly how the analysis of survival probabilities and default intensities of the previous sections can be directly applied to defaultable bond pricing.

The paper is concluded with some remarks on implementation and a summary and discussion of the results.

2. Preliminaries

2.1. **Notation.** For functions that use vectors $\mathbf{x} = (x_1, \ldots, x_N)$ as arguments we use the following notation if we replace the i-th component of $\mathbf{x}$ with y:

$$f(\mathbf{x}_{-i}, y) := f(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_N). \tag{1}$$

$\mathbf{1}$ is the vector $(1, \ldots, 1)$ and $\mathbf{0} = (0, \ldots, 0)$. Comparisons between vectors are meant component-wise. Vectors of functions $F_i : \mathbb{R} \to \mathbb{R}$ are written as

$$\mathbf{F}(\mathbf{x}) := (F_1(x_1), F_2(x_2) \ldots, F_N(x_N))). \tag{2}$$

Frequently, partial derivatives are written in index notation, i.e. $\frac{\partial}{\partial x_i} C() = C_{x_i}()$.For a stochastic process like $\lambda(\omega, t)$ we only write $\lambda(t)$, suppressing the dependence on ω. All filtered probability spaces in this paper are assumed to satisfy the usual conditions.

2.2. **Copula Functions.** In the following we want to model the joint dynamics of defaultable bond prices. This aim is twofold. On the one hand we have to model the default dynamics of a single obligor and on the other hand we have to model the dependence structure of the defaults between several obligors. In this section we present some basic tools for our dependency-modelling approach. Further details on dependency modelling and copula functions and the proofs of the propositions in this subsection can be found in the excellent books by Joe (1997), and Nelsen (1999).

Let $X_1, X_2, \ldots X_N$, denote real-valued random variables defined on the probability space $(\hat{\Omega}, \hat{\mathcal{A}}, P)$. Let $F_i(x)$ be the distribution function of X_i for $i \leq N$. For continuous distribution functions F_i note the following elementary fact:

Proposition 2.1.
Let $\hat{X}$ denote a continuous random variable with distribution function $\hat{F}$ then $Z = \hat{F}(\hat{X})$ has a uniform distribution on $[0, 1]$.

The dependence structure of real-valued random variables is completely described by their joint distribution. The joint distribution F of the random variables X_1, $X_2, \ldots X_N$ is[2]:

$$F(\mathbf{x}) = \mathbf{P}\left[\, X_1 \le x_1, X_2 \le x_2, \ldots, X_N \le x_N \,\right].$$

The basic idea of the analysis of dependency with copula functions is that the joint distribution function F can be separated into two parts. The first part is represented by the distribution functions of the random variables (marginals) and the other part is the dependence structure between the random variables which is described by the copula function.

We first present the definition of a copula and then describe several of its properties which will be used in this paper.

Definition 2.2.
A copula is any function $C : [0,1]^N \to [0,1]$ which has the following properties:

(1) $C(\mathbf{1}_{-i}, v_i) = v_i$ for all $i = 1, \ldots, N$, $v_i \in [0,1]$ and for every $\mathbf{v} \in [0,1]^N$, $C(\mathbf{v}) = 0$ if at least one coordinate of the vector $\mathbf{v}$ is 0;
(2) For all $\mathbf{a}, \mathbf{b} \in [0,1]^n$ with $\mathbf{a} \le \mathbf{b}$ the volume of the hypercube with corners $\mathbf{a}$ and $\mathbf{b}$ is positive, i.e. we have

$$\sum_{i_1=1}^{2} \sum_{i_2=1}^{2} \cdots \sum_{i_N=1}^{2} (-1)^{i_1+i_2+\ldots+i_N}\, C(v_{i_1}, v_{i_2}, \ldots, v_{i_N}) \;\ge\; 0$$

where $v_{j_1} = a_j$ and $v_{j_2} = b_j$ for all $j = 1, \ldots, N$.

This definition ensures that the copula can be used as a distribution function on $[0,1]^N$. The simplest example of a copula is the product copula which corresponds to the uniform distribution on $[0,1]^N$.

Example 2.3.
The N-dimensional product copula Π^N satisfies definition 2.2 and is given by:

$$\Pi^N(\mathbf{v}) = v_1 \cdot v_2 \cdot \ldots \cdot v_N.$$

We now state some properties of copula functions.

Proposition 2.4.
Let C be a N-dimensional copula. The copula C is non decreasing in each argument, i.e.

[2]Watch the notation: $F(\mathbf{x})$ is the joint distribution function at $\mathbf{x} = (x_1, \ldots, x_N)$ whereas $\mathbf{F}(\mathbf{x})$ is the vector of marginal probabilities $(F_1(x_1), \ldots, F_N(x_N))$. As $\mathbf{F}(\mathbf{x})$ is vector-valued, the difference will also be clear from the context.

if $\mathbf{v} \in [0,1]^N$ *then*

$$C(\mathbf{v}) \leq C(\mathbf{v}_{-j}, v'_j) \qquad \forall\, 1 \geq v'_j > v_j,\ \forall\, j \leq N.$$

(Fréchet-Hoeffding Bounds:) Let C *be a* N*-dimensional copula. Then for every* $\mathbf{v} \in [0,1]^N$*:*

$$W^N(\mathbf{v}) \leq C(\mathbf{v}) \leq M^N(\mathbf{v}),$$

with

$$W^N(\mathbf{v}) = \max(v_1 + v_2 + \ldots + v_N - N + 1, 0)$$
$$M^N(\mathbf{v}) = \min(v_1, v_2, \ldots, v_N).$$

The functions M^N are copulas for all $N \geq 2$ whereas the functions W^N are never copulas for $N > 2$. (W^2 is a copula.)

The basis of the analysis of multivariate dependence with copula functions is the following theorem of Sklar (1996):

Theorem 2.5 (Sklar)**.**
Let $X_1, \ldots, X_N$ *be random variables with marginal distribution functions* $F_1, F_2, \ldots, F_N$ *and joint distribution function* F*. Then there exists a* N *dimensional copula* C *such that for all* $\mathbf{x} \in \mathbb{R}^N$*,*

$$F(\mathbf{x}) = C(F_1(x_1), F_2(x_2), \ldots, F_N(x_N)) = C(\mathbf{F}(\mathbf{x})).$$

If $F_1, F_2, \ldots, F_N$ *are continuous, then* C *is unique. Otherwise* C *is uniquely determined on* $RanF_1 \times \ldots \times RanF_N$*, where* $RanF_i$ *denotes the range of* F_i *for* $i = 1, \ldots, N$*.*

Thus, for continuous multivariate distributions, the univariate margins and the dependence structure can be separated. The dependence structure is completely characterised by the copula C. We say that $X_1, X_2, \ldots X_N$ have a copula C, where C is given by theorem 2.5.

Copula functions are the most general way to view dependence of random variables. In particular this concept is much more general than linear correlation which often fails to capture important risks[3].

Theorem 2.5 also tells us that the copula of $\mathbf{X}$ is the distribution function of $\mathbf{F}(\mathbf{X})$, and that transformations of the X_i do not change the copula as long as all transformations are monotonous.

[3]See Embrechts et.al. (1999) for more details.

Independence of continuous random variables is also easily characterised with their copula: The $X_1, X_2, \ldots, X_N$ are independent if and only if the N-dimensional copula C of $X_1, X_2, \ldots, X_N$ is

$$C(\mathbf{F}(\mathbf{x})) \;=\; \Pi^N(\mathbf{F}(\mathbf{x})).$$

2.3. **Intensities and Survival Probabilities.** In this subsection we recall some basic properties of intensities and survival probabilities.

Definition 2.6. *Let $(\Omega^*, \mathcal{H}^*_{t\in[0,\overline{T}]}, Q^*)$ be a filtered probability space and $N^*(t) = \mathbf{1}_{\{\tau^* \le t\}}$ with τ^* a stoppping time. Let $M(t)$ denote the compensated process, thus*

$$M(t) = N^*(t) - A(t) \quad \forall t \in [0, \overline{T}]$$

follows a martingale under Q^*. N^* has a nonnegative intensity process $g(t)$ under Q^* if the compensator $A(t)$ can be represented by

$$A(t) = \int_0^t g(u)du \quad \forall t \in [0, \overline{T}].$$

The intensity is linked to the survival probability of the process N^*:

Proposition 2.7 (Aven)**.** *Let $\{h_n\}_{n=1}^{\infty}$ be a sequence which decreases to zero and let $Y_n(t), t \in [0, \overline{T}]$ be a measurable version of the process*

$$Y_n(t) := \frac{1}{h_n}\mathbf{E}\left[\, N^*(t+h_n) - N^*(t) \,|\, \mathcal{H}^*_t \,\right].$$

Assume that there are non-negative and measurable processes $g(t)$ and $y(t), t \in [0, \overline{T}]$ such that:

(i) for each t

$$\lim_{n\to\infty} Y_n(t) = g(t) \quad a.s.$$

(ii) for each t there exists for almost all $\omega \in \Omega$ an $n_0 = n_0(t, \omega)$ such that

$$|Y_n(s,\omega) - g(s,\omega)| \le y(s,\omega), \qquad \forall\, s \le t, n \ge n_0.$$

(iii)

$$\int_0^t y(s)\, ds \, < \infty, \; a.s., \quad t \in [0, \overline{T}]$$

then $N^*(t) - \int_0^t g(s)\, ds$ is a $\mathcal{H}^*_t$- martingale, and $\int_0^t g(s)\, ds$ is the compensator of $N^*(t)$.

From this follows directly our recipe for deriving the default intensities:

Lemma 2.8. *Let*

$$P(t,T) := \mathbf{Q}^* [\, \tau^* > T \mid \mathcal{H}_t^* \,]$$

denote the probability given $\mathcal{H}_t^*$ that the jump has not occured until T. Furthermore let $P(t,T)$ be differentiable from the right with respect to T at $T = t$, and let the difference quotients that approximate the derivative satisfy the assumptions of proposition 2.7. Then the intensity of the process N^* is given by:

$$\frac{dA(t)}{dt} = g(t) = -\frac{\partial}{\partial T}\bigg|_{T=t} P(t,T). \tag{3}$$

Thus, given certain regularity conditions, we can directly derive the default intensity if we have the survival probabilities.

3. Model Setup

The dependent-defaults model is built up in two steps: First we specify the stochastic model for individual defaults, and in a second step we introduce default dependency. In this section we describe the stochastic model for the defaults of the individual obligors. We consider an economy with $i = 1, \ldots, I$ obligors.

The basic probability space in which the model lives is $(\Omega, \widetilde{\mathcal{F}}, Q)$. Ω is assumed to be large enough to support all processes that are introduced. All subsequently introduced filtrations are subsets of $\widetilde{\mathcal{F}}$ and augmented by the zero-sets of $\widetilde{\mathcal{F}}$. The probability measure Q can – but need not – be a martingale measure for the specific filtrations considered.

First, we introduce the background filtration $(\mathcal{G}_t)_{t\in[0,\overline{T}]}$. This filtration represents information about the development of general market variables such as share prices, default-free interest-rates or exchange rates, and also all default-relevant information *except explicit information on the occurrence or non-occurrence of defaults*[4]. Thus, $(\mathcal{G}_t)_{t\in[0,\overline{T}]}$ can also contain information on credit spread movements and rating transitions (except for transitions to 'default'). To aid intuition we assume that $(\mathcal{G}_t)_{t\in[0,\overline{T}]}$ is generated by a stochastic process $X(t)$, the background process.

Definition 3.1. *The* background process $X(t)$ *is a m-dimensional stochastic process. We denote the filtration generated by $X(t)$ with $(\mathcal{G}_t)_{t\in[0,\overline{T}]}$, and $\mathcal{G} := \sigma(\bigcup_{t\in[0,\overline{T}]} \mathcal{G}_t)$.*

Defaults in this model are triggered as follows:

[4]Mathematically, $(\mathcal{G}_t)_{t\in[0,\overline{T}]}$ must be independent from all $\mathcal{U}_i$ (which will be introduced later).

Assumption 1 (Default Mechanism)**.**
The time of default τ_i *of obligor* $i = 1, \ldots, I$ *is the first time, when the default countdown process* $\gamma_i(t)$ *reaches the level of the trigger variable* U_i*:*

$$\tau_i := \inf\{t : \gamma_i(t) \leq U_i\}, \tag{4}$$

where:

- *(i) The* default trigger variables U_i*,* $i = 1, \ldots, I$ *are random variables on the unit interval* $[0, 1]$*.* $\sigma(U_i) =: \mathcal{U}_i$ *is the information generated by knowledge of* U_i*.*
- *(ii) The* pseudo default-intensity $\lambda_i(t)$ *is a nonnegative càdlàg stochastic process which is adapted to the filtration* $(\mathcal{G}_t)_{t\in[0,\overline{T}]}$ *of the background process. Denote* $\Lambda_i(t) := \int_0^t \lambda_i(s)ds$ *the integral of the intensity.*
- *(iii) The* default countdown process $\gamma_i(t)$ *is defined as*

$$\gamma_i(t) := \exp(-\int_0^t \lambda_i(u)\, du). \tag{5}$$

Furthermore we define the default and survival indicator processes $N_i(t) := \mathbf{1}_{\{\tau\leq t\}}$ *and* $I_i(t) := \mathbf{1}_{\{\tau_i>t\}}$*. Filtration* $(\mathcal{F}_t^i)_{t\in[0,\overline{T}]}$ *is the augmented filtration that is generated by* $N_i(t)$*.*

Default of obligor i is triggered as soon as the countdown process $\gamma_i(t)$ falls below the random trigger level U_i. The realisation of this trigger level remains unknown to the economy, only at default its value is revealed. The pseudo default intensity $\lambda_i(t)$ controls the speed of the countdown and thus the likelihood of an early default. The process $\lambda_i(t)$ is called *pseudo* default-intensity, because it coincides with the default intensity of obligor i in the "independence" case described below, or if information is restricted to information about obligor i alone. In general, it will *not* be the default intensity.

Usually, the introduction of the threshold levels U_i is not necessary when a totally inaccessible stopping time is defined, it is sufficient to specify the intensity process which gives all information on the distribution of the jump times. We introduced the U_i anyway because they will provide the handle to introduce default dependencies into the model. Furthermore, this setup directly shows how to efficiently simulate this model in a Monte-Carlo simulation: Draw the U_i, simulate the paths of λ_i (and thus γ_i) and determine the times of default.

This setup is very general. We deliberately did not restrict the dynamics of the stochastic processes of the pseudo default-intensities λ_i to give the reader the freedom to choose her favourite specification of an intensity model, based upon any of the models that were proposed in the literature. In particular, the dynamics of the pseudo-default intensities can be correlated or dependent themselves.

The modelling of the default time (4) is inspired by an observation in Lando (1998). Lando shows that the time of the first jump of a Cox process with intensity $\lambda(u)$ can be constructed as

$$\tau = \inf\{t : \int_0^t \lambda(u)\,du \geq E\}, \tag{6}$$

where E is a unit exponential random variable independent of λ. This setup is equivalent to taking logarithms of both sides of the inequality in equation (4) when U_i is uniformly distributed on $[0, 1]$. Almost all practical implementations of intensity-based models can be translated into a Cox-process framework.

In the analysis of this setup it is crucial to be very careful in the specification of the available information, because default probabilities will be different, and different information structures are plausible. We therefore introduce the following filtrations:

Definition 3.2. *(i) Filtration $(\tilde{\mathcal{H}}_t^i)_{t\in[0,\overline{T}]}$ contains information about the default or survival of obligor i up to time t, and* complete *information about the background process*

$$\tilde{\mathcal{H}}_t^i := \sigma\left(\mathcal{F}_t^i \cup \mathcal{G}\right).$$

(ii) Filtration $(\mathcal{H}_t^i)_{t\in[0,\overline{T}]}$ contains information about the default or survival of obligor i up to time t, and partial *information about the background process up to time t*

$$\mathcal{H}_t^i := \sigma\left(\mathcal{F}_t^i \cup \mathcal{G}_t\right).$$

(iii) Filtration $(\tilde{\mathcal{H}}_t)_{t\in[0,\overline{T}]}$ reflects information about the defaults of all *obligors until t, and* complete *information about the background process:*

$$\tilde{\mathcal{H}}_t = \sigma\left(\bigcup_{i=1}^{I} \tilde{\mathcal{H}}_t^i\right).$$

(iv) Filtration $(\mathcal{H}_t)_{t\in[0,\overline{T}]}$ is the equivalent of $(\tilde{\mathcal{H}}_t)_{t\in[0,\overline{T}]}$, but with the information on the background process restricted to $[0, t]$

$$\mathcal{H}_t = \sigma\left(\bigcup_{i=1}^{I} \mathcal{H}_t^i\right).$$

(v) Filtration $(\mathcal{F}_t)_{t\in[0,\overline{T}]}$ contains only default information of all I obligors up to time t

$$(\mathcal{F}_t)_{t\in[0,\overline{T}]} = \sigma\left(\bigcup_{i=1}^{I} \mathcal{F}_t^i\right).$$

Definition 3.2 contains all possible permutations of information sets:

- Filtrations indexed with i contain only information on *one* obligor.
- Filtrations with a tilde contain full information on the background process X. This is unrealistic, but useful in the mathematical derivations.
- Filtrations without a tilde only contain information that is available at time t. In particular, $\mathcal{H}_t$ represents all available information in the economy at time t. Filtration $\mathcal{H}_t^i$ represents all available information if we restrict our attention only on obligor i and the background process.

These filtrations enable us to model the intensity of the default process of one obligor independent of the information about the default behaviour of the remaining $I-1$ obligors. As shown in the following sections the default process N_i has a different intensity under the filtration $(\mathcal{H}_t^i)_{t\in[0,\overline{T}]}$ than under $(\mathcal{H}_t)_{t\in[0,\overline{T}]}$. Accordingly, we need to define different survival probabilities.

Definition 3.3 (Survival Probabilities)**.**
For each of obligor i, $i = 1, \ldots, I$ define

(1) the survival probability until T given the information $(\mathcal{H}_t^i)_{t\in[0,\overline{T}]}$:

$$P_i'(t,T) := E^Q[I_i(T)|\mathcal{H}_t^i],$$

(2) the survival probability until T given the information $(\tilde{\mathcal{H}}_t^i)_{t\in[0,\overline{T}]}$:

$$\tilde{P}_i'(t,T) := E^Q[I_i(T)|\tilde{\mathcal{H}}_t^i].$$

The dashes indicate that these probabilities are survival probabilities under the restriction to the default history of obligor i alone, i.e. ignoring the other obligors.

Next we make our first assumption about the distributions of the U_i, $i = 1, \ldots, I$:

Assumption 2.
For all $i = 1, \ldots, I$, the default threshold U_i is uniformly distributed on $[0,1]$ under $(Q, \mathcal{H}_0^i)$, and $\mathcal{U}_i$ is independent from $\mathcal{G}_\infty$ under Q.

Under this assumption only the *marginal* distribution of the U_i is prescribed, because the filtration $(\mathcal{H}_t^i)_{t\in[0,\overline{T}]}$ does not contain information about the other U_j. Now we can calculate the survival probabilities:

Proposition 3.4.
Under assumption 2 and given $\tau_i > t$ the survival probabilities are:

$$\tilde{P}'_i(t,T) = \frac{\gamma_i(T)}{\gamma_i(t)} = e^{-\int_t^T \lambda_i(s)ds} \tag{7}$$

$$P'_i(t,T) = \mathbf{E}^Q \left[\frac{\gamma_i(T)}{\gamma_i(t)} \,\middle|\, \mathcal{H}_t^i \right] = \mathbf{E}^Q \left[e^{-\int_t^T \lambda_i(s)ds} \,\middle|\, \mathcal{H}_t^i \right]. \tag{8}$$

Proof. Equation (7):

$$\tilde{P}'_i(t,T) = \mathbf{Q}\left[\tau_i > T \,\middle|\, \tilde{\mathcal{H}}_t^i \wedge \{\tau_i > t\} \right] = \mathbf{Q}\left[\gamma_i(T) > U_i \,\middle|\, \tilde{\mathcal{H}}_t^i \wedge \{\tau_i > t\} \right].$$

From $\tau_i > t$ we know that $U_i < \gamma_i(t)$, thus U_i is uniformly distributed on $[0, \gamma_i(t)]$. Therefore

$$\tilde{P}'_i(t,T) = \frac{\gamma_i(T)}{\gamma_i(t)}.$$

Equation (8): From $\mathcal{H}_t^i \subset \tilde{\mathcal{H}}_t^i$ follows for all random variables X:

$$\mathbf{E}\left[X \mid \mathcal{H}_t^i \right] = \mathbf{E}\left[\mathbf{E}\left[X \,\middle|\, \tilde{\mathcal{H}}_t^i \right] \,\middle|\, \mathcal{H}_t^i \right].$$

□

As stated in lemma 2.8 we can characterize the intensity of the default process through the survival probabilities.

Proposition 3.5.
The intensity of the process $N_i(t)$ under the filtration $(\mathcal{H}_t^i)_{t\in[0,\overline{T}]}$ is

$$-\frac{\partial}{\partial T} P'_i(t,T)\bigg|_{T=t} = \mathbf{1}_{\{\tau^i > t\}} \lambda_i(t).$$

Proof. By differentiation. We can differentiate under the integral sign since the random variable $\gamma_i(T)/\gamma_i(t)$ is bounded by 1 and therefore uniformly integrable. □

Proposition 3.5 shows, that λ_i is indeed the default intensity of obligor i, *if only this obligor and the general state of the economy is observed.* This proves our claim that by reducing the information set to $(\mathcal{H}_t^i)_{t\in[0,\overline{T}]}$ the model reduces to a standard default risk model of the intensity-type for a single obligor.

Dependence between the defaults of the I obligors is introduced through the specification of the joint distribution of the random variables $U_1, U_2, \dots, U_I$:

Assumption 3.
Under $(\mathcal{H}_0, Q)$ the I-dimensional vector $\mathbf{U} = (U_1, \dots, U_I)$ is distributed according to the I-dimensional copula

$$C(\mathbf{u}).$$

$\mathbf{U}$ *is independent from $\mathcal{G}_\infty$. Furthermore, C is twice continuously differentiable.*

Because the marginal distributions of a copula function are uniform, this assumption is consistent with assumption 2. We could also have used a general distribution function on $[0,1]^I$ but that would have meant giving up the following advantage of uniform margins: Introducing the joint default distribution via the copula function $C(\mathbf{u})$ does *not* change the individual default probabilities if we ignore the other obligors. It also does not change the default probabilities as seen from $t = 0$. At later times and given information on the default- and survival behaviour of the other obligors, this will be different.

It should be pointed out that — besides the dependency between the trigger levels U_i — the model can also accommodate correlations between the pseudo default-intensities $\lambda_i(t)$. As mentioned in the introduction, this alone would not generate default correlations of a realistic size, but as an augmentation of our modelling approach it is useful.

Remark 4.1. *(i) For every differentiable copula function we have*

$$0 \leq \frac{\partial}{\partial v_i} C(\mathbf{v}) \leq 1 \quad \forall v_i \in [0,1],\ i = 1 \dots, I.$$

(ii) In the single-obligor case, under $(\tilde{\mathcal{H}}_t)_{t\in[0,\overline{T}]}$ the distribution function of the time τ_i of the default of obligor i is $F_i(t) = 1 - \gamma_i(t)$.
A similar result now holds in the multi-obligor case: Under $(\tilde{\mathcal{H}}_t)_{t\in[0,\overline{T}]}$ the joint distribution function of the times $\boldsymbol{\tau} = (\tau_1, \dots, \tau_I)$ of default is

$$\mathbf{Q}\left[\, \boldsymbol{\tau} \geq \mathbf{t} \,\middle|\, \tilde{\mathcal{H}}_0 \right] = \bar{F}(\mathbf{t}) = C(\gamma_1(t_1), \dots, \gamma_I(t_I)) = C(\boldsymbol{\gamma}(\mathbf{t})). \tag{9}$$

As usual, taking the expectation of (9) yields the corresponding distribution function under $(\mathcal{H}_t)_{t\in[0,\overline{T}]}$.

We now introduce survival probabilities which are conditioned on events included in the filtrations $(\tilde{\mathcal{H}}_t)_{t\in[0,\overline{T}]}$ and $(\mathcal{H}_t)_{t\in[0,\overline{T}]}$. In particular we consider two classes of events. In the first case all obligors have survived until time t and in the other case one obligor has undergone a default at time t.

4.1. **Probabilities before defaults.**

Definition 4.2.
Survival Probabilities *For each obligor i, $i \leq I$ we define*

(1) *the survival probability given the information $(\mathcal{H}_t)_{t\in[0,\overline{T}]}$:*

$$P_i(t,T) := \mathbf{E}^Q\left[\, I_i(T)\,|\,\mathcal{H}_t\,\right],$$

(2) *the survival probability given the information $(\tilde{\mathcal{H}}_t)_{t\in[0,\overline{T}]}$:*

$$\tilde{P}_i(t,T) := \mathbf{E}^Q\left[\, I_i(T)\,\middle|\,\tilde{\mathcal{H}}_t\,\right].$$

(3) *the default intensity $h_i(t)$ for information $(\mathcal{H}_t)_{t\in[0,\overline{T}]}$ and $\tilde{h}_i(t)$ for $(\tilde{\mathcal{H}}_t)_{t\in[0,\overline{T}]}$.*

The following is the analogous result to proposition 3.4.

Proposition 4.3.
If no obligor has defaulted until t the individual survival probabilities and default intensities are

$$\tilde{P}_i(t,T) = \frac{C(\boldsymbol{\gamma}_{-i}(t),\gamma_i(T))}{C(\boldsymbol{\gamma}(t))} \tag{10}$$

$$P_i(t,T) = \mathbf{E}^Q\left[\,\frac{C(\boldsymbol{\gamma}_{-i}(t),\gamma_i(T))}{C(\boldsymbol{\gamma}(t))}\,\middle|\,\mathcal{H}_t\right] \tag{11}$$

$$h_i(t) = \tilde{h}_i(t) = \lambda_i(t)\gamma_i(t)\frac{\frac{\partial}{\partial x_i}C(\boldsymbol{\gamma}(t))}{C(\boldsymbol{\gamma}(t))} = \lambda_i(t)\gamma_i(t)\frac{\partial}{\partial x_i}\ln C(\boldsymbol{\gamma}(t)). \tag{12}$$

Proof. Apply Bayes' rule (for the conditioning on survival until t) and assumption 3. Differentiate to reach the default intensity. As $\tilde{h}_i(t)$ only depends on quantities that are $\mathcal{H}_t$-measurable, the default intensities under $(\mathcal{H}_t)_{t\in[0,\overline{T}]}$ and $(\tilde{\mathcal{H}}_t)_{t\in[0,\overline{T}]}$ coincide. (This is no surprise given the local character of the default intensity.) □

The intensity $h_i(t)$ still depends on the one-obligor intensity $\lambda_i(t)$ from proposition 3.5, but it is modified. This modification reflects the fact, that under the filtration $(\mathcal{H}_t)_{t\in[0,\overline{T}]}$ we are able to observe more information on the default likelihood of obligor i than under filtration $(\mathcal{H}^i_t)_{t\in[0,\overline{T}]}$. The additional information is the information, that the other obligors have not defaulted yet, it is information on U_i than can be inferred from $\mathbf{U}_{-i} < \boldsymbol{\gamma}_{-i}(t)$.

If the U_i are independent, observing the survival and defaults of the other obligors does not convey information on obligor i. If the copula in proposition 4.3 is the independence copula $\Pi^I()$ then

$$h_i(t) = \lambda_i(t) \qquad \forall\, i \leq I. \tag{13}$$

The intensities $h_i(t)$ and $\lambda_i(t)$ also coincide at $t = 0$: Here the partial derivatives of C coincide with the marginal derivatives $C_{x_i}(\mathbf{1}) = \mathbf{1}$.

It is easy to check that $\lambda_i(t)$ is recovered as intensity if we artificially restrict our information to $\mathcal{H}_t^i$. Thus

$$\lambda_i(t) = \mathbf{E}^Q \left[\, h_i(t) \, \middle| \, (\mathcal{H}_t^i)_{t \in [0,\overline{T}]} \right] \qquad \forall\, i \leq I, \tag{14}$$

a single-obligor default risk model can be viewed as the projection of a multi-obligor default risk model onto the one-obligor world. The same results apply to the survival probabilities $P_i(t,T)$ and $P_i'(t,T)$.

4.2. **Probabilities after defaults.** If one or more of the obligors have already defaulted, we move to a *conditional* distribution function. If obligor i defaults at time t, we have to use the conditional distribution of the $\mathbf{U}$ from that time onwards, conditional on $U_i = \gamma_i(t)$. This function is given in the following lemma:

Lemma 4.4. *Assume the times of default of $k < I$ obligors are known at time t. W.l.o.g. we assume that these are the first k obligors. If C is sufficiently differentiable, the conditional distribution function of $\boldsymbol{\tau}$ is*

$$\begin{aligned} &\mathbf{Q}\left[\, \boldsymbol{\tau} \geq \mathbf{T} \, \middle| \, \tilde{\mathcal{H}}_t \wedge \{\tau_i = t_i \; for \; 1 \leq i \leq k\} \wedge \{\tau_j > t \; for \; k < j \leq I\} \right] \\ &= \frac{\frac{\partial^k}{\partial x_1 \cdots \partial x_k} C(\gamma_1(t_1), \ldots, \gamma_k(t_k), \; \gamma_{k+1}(T_{k+1}), \ldots, \gamma_I(T_I))}{\frac{\partial^k}{\partial x_1 \cdots \partial x_k} C(\gamma_1(t_1), \ldots, \gamma_k(t_k), \; \gamma_{k+1}(t), \ldots, \gamma_I(t))}, \end{aligned} \tag{15}$$

and

$$\begin{aligned} &\mathbf{Q}\left[\, \boldsymbol{\tau} \geq \mathbf{T} \, | \, \mathcal{H}_t \wedge \{\tau_i = t_i \; for \; 1 \leq i \leq k\} \wedge \{\tau_j > t \; for \; k < j \leq I\} \right] \\ &= \frac{\mathbf{E}^Q \left[\, \frac{\partial^k}{\partial x_1 \cdots \partial x_k} C(\gamma_1(t_1), \ldots, \gamma_k(t_k), \; \gamma_{k+1}(T_{k+1}), \ldots, \gamma_I(T_I)) \, \middle| \, \mathcal{H}_t \right]}{\frac{\partial^k}{\partial x_1 \cdots \partial x_k} C(\gamma_1(t_1), \ldots, \gamma_k(t_k), \; \gamma_{k+1}(t), \ldots, \gamma_I(t))}, \end{aligned} \tag{16}$$

where $T_i > \tau_i$ for $i \leq k$.

Apart from a different form for the distribution function of the U_i the situation is now identical to the situation before any defaults in the previous subsection. Thus, the survival probabilities and default intensities of the remaining obligors are reached by substituting the conditional distribution function (15) as distribution function in proposition 4.3.

The conditional distribution function is actually not a copula function but just a general distribution function on the unit hypercube (with non-uniform marginals): To reach a *copula* distribution function again we need to transform the γ_i with the respective marginal distribution functions. This is not really necessary because the uniform marginals were only used to conveniently reduce the model to the one-obligor case.

4.3. **Probabilities *at* defaults.** Because we assumed C to be differentiable, joint defaults at *exactly* the same time have probability zero in this model. Thus, fundamentally there are only two types of points in time: Points when no default occurs, and points in time when a default happens. The survival probabilities and default intensities in the standard case (no default) were given in the previous proposition. Now we analyze what happens to the survival probabilities of the other obligors *at* the time of default of an obligor j. At this point in time the distribution of default times changes discretely to the conditional distribution function according to lemma 4.4.

Definition 4.5.
Survival Probabilities given default *For each obligors i and j, $i, j = 1, \dots, I$ define*

(1) the survival probability of i given the information $(\mathcal{H}_t)_{t\in[0,\overline{T}]}$ and given a default of obligor j at t:

$$P_i^{-j}(t,T) := \mathbf{E}^Q\left[\, I_i(T) \,|\, \mathcal{H}_t \wedge \{\tau_j = t\}\,\right],$$

(2) the survival probability of i given the information $(\tilde{\mathcal{H}}_t)_{t\in[0,\overline{T}]}$ and given a default of obligor j at t:

$$\tilde{P}_i^{-j}(t,T) := \mathbf{E}^Q\left[\, I_i(T) \,\middle|\, \tilde{\mathcal{H}}_t \wedge \{\tau_j = t\}\,\right],$$

Using equation (15) these survival probabilities take the following values: (We will now use subscript notation for partial derivatives of C, i.e. $C_{x_i} = \frac{\partial}{\partial x_i} C$ etc.)

Proposition 4.6.
If obligor j has defaulted at t and all other obligors are still alive at t, the individual

survival probabilities and default intensities are for obligor i

$$\tilde{P}_i^{-j}(t,T) = \frac{C_{x_j}(\boldsymbol{\gamma}_{-\mathbf{i}}(t), \gamma_i(T))}{C_{x_j}(\boldsymbol{\gamma}(t))} \tag{17}$$

$$P_i^{-j}(t,T) = \mathbf{E}^Q\left[\frac{C_{x_j}(\boldsymbol{\gamma}_{-\mathbf{i}}(t), \gamma_i(T))}{C_{x_j}(\boldsymbol{\gamma}(t))} \,\middle|\, \mathcal{H}_t \right] \tag{18}$$

$$h_i^{-j}(t) = \lambda_i(t)\gamma_i(t)\frac{C_{x_i x_j}(\boldsymbol{\gamma}(t))}{C_{x_j}(\boldsymbol{\gamma}(t))}. \tag{19}$$

Comparing this to the survival probabilities in proposition 4.3, we see that the survival probabilities after a default of obligor j now use the conditional distribution function of the U_i. The partial derivative is taken w.r.t. the defaulted obligor j in the copula function, and the j-th argument of the copula is set to $\gamma_j(t)$ at $t = \tau_j$ (and it also remains at $\gamma_j(\tau_j)$ at later times).

Except in the independence case, the new default intensity of i will not be equal to the default intensity of i before the default of j. The change to the conditional distribution will result in a jump in the default intensity and in the survival probabilities. This effect is quantified in the next subsection.

4.4. **Dynamics of default intensities.** If the pseudo-default intensities λ_i follow diffusion processes, the dynamics of the default intensities h_i follow directly from Itô's lemma:

Proposition 4.7.
If no default has happened until time t, the dynamics of the default intensity h_i are given by

$$dh_i = \frac{C_{x_i}}{C} \cdot \gamma_i\lambda_i \cdot \left[\left(\frac{d\lambda_i}{\lambda_i} - \lambda_i\, dt\right) - \sum_{j=1}^{I} \left(\frac{C_{x_i x_j}}{C_{x_i}} - \frac{C_{x_j}}{C}\right)\gamma_j\lambda_j\, dt \right], \tag{20}$$

if there is no default at t, and a jump of

$$\Delta h_i = \lambda_i\gamma_i\frac{C_{x_i}}{C}\left[\frac{C_{x_j x_i}\, C}{C_{x_j} C_{x_i}} - 1\right] \tag{21}$$

if obligor $j \neq i$ defaults at time t. (Suppressing the arguments t and $\boldsymbol{\gamma}(t)$.)
This can be re-written as:

(22)

$$\frac{dh_i}{h_i} = \frac{d\lambda_i}{\lambda_i} + (h_i\big(1 - \frac{C_{x_i x_i}\, C}{C_{x_i}^2}\big) - \lambda_i)dt - dN_i + \sum_{j=1,\, j\neq i}^{I} \left(\frac{C_{x_i x_j}\, C}{C_{x_i}\, C_{x_j}} - 1\right)(dN_j - h_j dt).$$

Equation (22) has the following interpretation: First, the default intensity h_i of obligor i is driven by the pseudo default intensity λ_i. This is the only source of diffusion risk in h_i, in particular we will have the same volatility for λ_i and h_i. The next term is a correction term for the fact that in general $\lambda_i \neq h_i$ and that C is not linear in x_i. Then follows $-dN_i$ which sets h_i to zero after the default of obligor i.

The influence of potential defaults of other obligors is contained in the summation term in equation (22). This term is of the form of a compensated jump process: There is a large jump component (triggered by $dN_j = 1$) at a default of obligor j.

The sign of the jump is determined by the dependence between the default times of i and j. A common measure for the dependence between two random variables X and Y is *Kendall's tau*. It can be written as

$$\rho_\tau = 2 \int_{[0,1]^2} C(u,v)C_{uv}(u,v) - C_u(u,v)C_v(u,v)du\,dv, \tag{23}$$

where $C(u,v)$ is the copula function of X and Y. (See Durrleman et.al. (2000).) From equation (23) follows directly, that a positive sign of the jump terms in proposition 4.7 corresponds to a positive contribution to Kendall's tau, and thus to a locally positive dependence between U_i and U_j.

If there is positive (negative) dependence between the defaults of obligors i and j then the survival probability of i will drop (increase) at a *default* of j, for independence it remains unchanged. If on the other hand obligor j *survives* over the infinitesimal interval $[t, t+dt]$ then this will have the opposite effect of a default on the intensity of i. Survival of j means that a default of i is less (more) likely if i and j are positively (negatively) dependent.

5. Examples

5.1. **Archimedean Copulas.** A popular class of copula functions are the Archimedean copula functions which have the following representation:

Definition 5.1 (Archimedean Copulas).
An Archimedean copula function is a copula function which has the following representation:

$$C(\mathbf{x}) = \phi^{[-1]}\Big(\sum_{i=1}^{I} \phi(x_i)\Big).$$

The function $\phi(\cdot)$ is called the generator *of the copula, $\phi^{[-1]}$ denotes the inverse function of ϕ.*

The generator has the following properties[5]: $\phi : [0,1] \to \mathbb{R}^+ \cup \{+\infty\}$, ϕ is invertible, $\phi'(x) < 0$, $\phi''(x) > 0$. Archimedean copulas are capable of reproducing a large variety of possible dependence structures.

It is straightforward to check that for an Archimedean copula C with generator ϕ the partial derivatives with respect to x_i and x_j, $i, j \le I$ are given by

$$C_{x_i}(\mathbf{x}) = \frac{\phi'(x_i)}{\phi'(C(\mathbf{x}))} \qquad C_{x_i x_j}(\mathbf{x}) = -\phi'(x_i)\phi'(x_j)\frac{\phi''(C(\mathbf{x}))}{(\phi'(C(\mathbf{x})))^3}.$$

Thus we reach the following proposition:

Proposition 5.2. *If the distribution function C of the U_i is an Archimedean copula function with generator ϕ, then for any obligor $i \le I$*

(a) the default intensity before any defaults is

$$h_i(t) = \frac{\phi'(\gamma_i)}{C(\boldsymbol{\gamma})\phi'(C(\boldsymbol{\gamma}))}\gamma_i\lambda_i,$$

(b) at default $t = \tau_j$ of obligor $j \ne i$ the default intensity changes by a factor

$$h_i^{-j}(t) = h_i\left(-\frac{C(\boldsymbol{\gamma})\phi''(C(\boldsymbol{\gamma}))}{\phi'(C(\boldsymbol{\gamma}))}\right),$$

(c) the dynamics of h_i are

$$\frac{dh_i}{h_i} = \frac{d\lambda_i}{\lambda_i} - \lambda_i dt - \left(-\frac{C\phi''(C)}{\phi'(C)}\right)dN_i + \sum_{j=1}^{I}\left(\left(-\frac{C\phi''(C)}{\phi'(C)}\right) - 1\right)(dN_j - h_j dt).$$

In particular, the symmetric nature of the Archimedean copula functions has the effect that at a default of an obligor j, the default intensities of *all other obligors change by the same factor.* The default risk of obligor i depends on only two components: The individual default risk which is represented by $\phi'(\gamma_i)$, and the default dependency which can be summarised by $C(\boldsymbol{\gamma})$.

5.2. **Gumbel and Clayton Copulae.**

Definition 5.3 (Gumbel Copula)**.**
With $\phi(x) = (-\ln(x))^\theta$ for $\theta \in [1,\infty)$ in definition 5.1 we reach the one-parameter Gumbel

[5]These conditions are necessary, but not sufficient for ϕ to be a generator of a I-copula.

copula:

$$C(\mathbf{x}) = \exp\left\{-\left[\sum_{i=1}^{I}(-\ln x_i)^{\theta}\right]^{\frac{1}{\theta}}\right\}.$$

The generator of the Clayton copula is $\phi(x) = (x^{-\alpha} - 1)/\alpha$ for $\alpha > 0$. Then

$$C(\mathbf{x}) = \left(1 - I + \sum_{i=1}^{I} x_i^{-\alpha}\right)^{-\frac{1}{\alpha}}.$$

The interesting quantities in proposition 5.2 are the default intensity h_i and the jump in the default intensity of i if j defaults. These are for the Gumbel copula

$$h_i(t) = \frac{\phi'(\gamma_i)}{C(\boldsymbol{\gamma})\phi'(C(\boldsymbol{\gamma}))}\gamma_i\lambda_i, \qquad = \left(\frac{\Lambda_i}{\|\boldsymbol{\Lambda}\|_\theta}\right)^{\theta-1}\lambda_i \tag{24}$$

$$h_i^{-j}(t) = \left(-\frac{C(\boldsymbol{\gamma})\phi''(C(\boldsymbol{\gamma}))}{\phi'(C(\boldsymbol{\gamma}))}\right)h_i, \qquad = \left(1 + \frac{(\theta-1)}{\|\boldsymbol{\Lambda}\|_\theta}\right)h_i \tag{25}$$

and for the Clayton copula

$$h_i(t) = \frac{\phi'(\gamma_i)}{C(\boldsymbol{\gamma})\phi'(C(\boldsymbol{\gamma}))}\gamma_i\lambda_i, \qquad = \left(\frac{C(\boldsymbol{\gamma})}{\gamma_i}\right)^{\alpha}\lambda_i \tag{26}$$

$$h_i^{-j}(t) = \left(-\frac{C(\boldsymbol{\gamma})\phi''(C(\boldsymbol{\gamma}))}{\phi'(C(\boldsymbol{\gamma}))}\right)h_i \qquad = (1+\alpha)h_i. \tag{27}$$

where we wrote $\|\mathbf{x}\|_\theta := \left(\sum_{i=1}^{I}|x_i|^\theta\right)^{\frac{1}{\theta}}$ for the θ-norm in $\mathbb{R}^I$ and $\Lambda_i(t) := \int_0^t \lambda_i(s)ds$.

With the Clayton copula we regain a feature of the Davis/Lo (1999a; 1999b) model: A jump in the credit spread by a constant factor $(1+\alpha)$ at a default of another obligor. The copula approach yields several new insights into that model: The corresponding copula is the Clayton copula function, and we can directly give the distribution of the default times. Furthermore, we now can transform the model to a fixed time-horizon model which greatly facilitates the estimation of the parameter α. There are some differences in the details of the model, e.g. Davis and Lo suggest to let the increased hazard rate return to pre-default levels after an exponentially distributed crisis period, while here there is the drift correction to the default intensity which tends to reduce the default intensities continuously as long as no default happens.

For the Gumbel copula, the intensity $h_i(t)$ depends on the factor $\frac{\Lambda_i(t)}{\|\boldsymbol{\Lambda}(t)\|_\theta}$, which represents the dependence structure of the default times. This is the i-th component of the cumulative intensity-vector $\boldsymbol{\Lambda}$, normalized to one in the θ-norm. For constant pseudo-intensities λ_i this factor will be constant, thus making the model time-invariant before defaults. In contrast to this, the jump factor when a default happens is not constant but approaches 1 as time proceeds, thus preventing the default intensities from increasing too much.

5.3. **Estimation, Gaussian Copula and Credit Metrics.** It is well-known that portfolio default-risk models like CreditMetrics have a Gaussian copula structure if they are based upon the Merton (1974) firm's value model. These models are usually calibrated to the historical default experience of a country or industry group over a given (e.g. one-year) time horizon. It may be argued that a copula of default events over a one-year horizon would look different from a copula of default-times as it is used in this model. This is not the case because copula functions are invariant under monotonic increasing transformations of the marginals (see e.g. Li, Credit Metrics Monitor or Joe (1997)). If the transformation is monotonically decreasing, the copula of the transformed variables is equal to the *survival copula* of the original variables. For many important cases (e.g. Gaussian copula, $t-$copula), the survival copula is identical to the original copula.

We now show, that we can indeed directly derive the copula $C(\cdot)$ that we use in the model from historically observed default frequencies.

The indicator function of survival beyond the time-horizon T can be approximated arbitrarily well by continuously differentiable function, for example cumulative normal distribution functions:

$$\mathbf{1}_{\{\tau_i > T_i\}} = \lim_{n\to\infty} g_{n,T_i}(\tau_i) =: \lim_{n\to\infty} N\left(\frac{\tau_i - T_i}{n}\right).$$

For all n, $g_{n,T_i}(\cdot)$ is a strictly monotonically increasing function. Call $C_{g,n,\mathbf{T}}(\cdot)$ the copula function of the random variables $Y_{i,n} := g_{n,T_i}(\tau_i)$. By the invariance of copulae under strictly monotonous transformations, $C_{g,n,\mathbf{T}}(\cdot)$ will be equal to the copula of the default times $C_\tau(\cdot)$ for all n and for all choices of transformation function g and time horizons T_i.

Analogously, if instead of *survival indicators*, we construct *default indicators*, we reach the *survival copula* of the default times. Again this result will hold independently of n or the approximation functions g or time horizons T_i. Thus, the copula will remain valid even if the limit is taken as $n \to \infty$.

On the other hand, by construction, the copula function $C(\cdot)$ of the model is the copula function of yet another transformation of the default times. By (9), the survival probabilities are given by

$$\mathbf{Q}\left[\,\boldsymbol{\tau} \geq \mathbf{t}\;\middle|\;\tilde{\mathcal{H}}_0\right] = \bar{F}(\mathbf{t}) = C(\gamma_1(t_1), \ldots, \gamma_I(t_I)) = C(\boldsymbol{\gamma}(\mathbf{t})).$$

Thus, *the model copula $C(\cdot)$ is the survival copula of the default-time copula $C_\tau(\cdot)$.*

In the limit as $n \to \infty$, the approximated default indicator functions will approach the default indicator functions themselves. We can therefore use a fixed-horizon default copula function in our model, for example the Gaussian default copula function of the CreditMetrics model.

The Gaussian copula is not in the class of Archimedean copula functions and the terms in propositions 4.3, 4.4 and 4.6 do not simplify nicely, so we refrain from stating them here. Sampling from a Gaussian copula is very easy, though, so this should not be an obstacle to the numerical implementation of the model.

If a CreditMetrics (or similar) Gaussian correlation matrix R for the values of the assets of the obligors is given, random samples from the corresponding copula function can be drawn according to the following algorithm:

(1) Draw $\mathbf{X}$, a vector of I standard normal random variates which have the correlation matrix R
(2) $Y_i := N^{[-1]}(X_i)$, $i \leq I$ have the corresponding Gaussian copula as distribution. ($N^{[-1]}$ is the inverse of the standard normal distribution function.)

The time-transformation argument applies to all copula functions. Therefore, the choice of an appropriate copula and the estimation of its parameters can all be achieved by analysing default frequencies over a given time-horizon. Alternatively, one can try to fit the jump sizes of the spreads at a given default to historical experience or personal intuition. Even if one does not want to calibrate the model to the jump sizes at default, it is still advisable to use them for a plausibility check.

6. Defaultable bond pricing

In this section we describe briefly how the copula-approach can be used to link several single-name intensity models with the default copula, and how to calibrate the resulting model. To simplify the exposition we will only consider the pricing of defaultable zero coupon bonds under zero recovery, for a serious implementation we recommend the 'recovery of par' model as it is described in Duffie (1998a) and Schönbucher (1999). In these models, prices of defaultable zero-coupon bonds with zero recovery are the central ingredient to which the model is calibrated.

Assume that we have I independent intensity-based models that are calibrated to the term structure of default risk of our obligors. I.e. we have found dynamics for the processes $\lambda_i(t)$ such that the fit to the initial term structure of defaultable bond prices $\overline{B}_i(0,T)$ is achieved for all $T \geq 0$, $i \leq I$:

$$\overline{B}_i(0,T) = \mathbf{E}^Q \left[e^{-\int_0^T r(s)ds} \mathbf{1}_{\{\tau_i > T\}} \,\middle|\, \mathcal{H}_0^i \right] = \mathbf{E}^Q \left[e^{-\int_0^T r(s)ds} e^{-\int_0^T \lambda_i(s)ds} \,\middle|\, \mathcal{H}_0^i \right]. \tag{28}$$

For simplicity, these models are set in the Cox-process framework of section 3.

To embed these models into a copula model with dependent defaults we define the background filtration $\mathcal{G}_\infty$ is the union of the individual background filtrations and the filtration generated by the interest-rate process. As described earlier, the defaults are generated with the default countdown processes, with copula-dependent trigger levels U_i.

It remains to be shown that in this general model the initial term structures of defaultable bond prices remain indeed calibrated. This follows from

$$\overline{B}_i(0,T) \stackrel{?}{=} \mathbf{E}^Q \left[e^{-\int_0^T r(s)ds} \mathbf{1}_{\{\tau_i > T\}} \,\middle|\, \mathcal{H}_0 \right]$$

using iterated expectations yields

$$= \mathbf{E}^Q \left[\mathbf{E}^Q \left[e^{-\int_0^T r(s)ds} \mathbf{1}_{\{\tau_i > T\}} \,\middle|\, \tilde{\mathcal{H}}_0^i \right] \,\middle|\, \mathcal{H}_0 \right]$$

the discount factor is measurable with respect to the inner filtration, so

$$= \mathbf{E}^Q \left[e^{-\int_0^T r(s)ds} \mathbf{E}^Q \left[\mathbf{1}_{\{\tau_i > T\}} \,\middle|\, \tilde{\mathcal{H}}_0^i \right] \,\middle|\, \mathcal{H}_0 \right]$$

the survival probability follows from the results earlier in the paper

$$= \mathbf{E}^Q \left[e^{-\int_0^T r(s)ds} \gamma_i(T) \,\middle|\, \mathcal{H}_0 \right]$$

and this expectation just depends on the background processes and not the copula structure, therefore we reach

$$= \overline{B}_i(0,T),$$

as claimed.

Thus, the calibration of the dynamics of the pseudo-default intensity processes λ_i to reproduce given term structures of defaultable bond prices will in fact mean that the copula-dependent default risk model will also be calibrated to these initial term structures.

This connection is only valid at $t = 0$, because it uses the uniform marginal distributions of the copula function C. If the term structure of defaultable bond prices is to be recovered at a later time, then one must use the conditional distribution of the U_i which will not have uniform margins. This will not be a problem because the probabilities are directly given through the copula.

7. Conclusion

The setup of the model directly gives the strategy for an implementation via Monte-Carlo simulation: First, the default trigger-levels are drawn according to the copula distribution function, then the intensities and interest-rate processes are drawn independently.

It is also possible to invert the order of the simulation procedure: First simulate the paths of interest-rates and intensities, and then simulate the trigger levels and times of default. As the trigger levels are independent from the rest of the model, the second step can be repeated several times. This can lead to a significant saving in simulation time, because the realisation of the trigger levels U_i usually causes the largest variance in the sample, while the simulation of the paths of intensities and rates costs most computer time. This is subject to further research.

The copula model for default dependency in this paper achieves several goals: It builds on existing intensity-based models for individual default risk, its calibration to individual term structures of credit spreads and credit spread volatilities is straightforward, for the calibration of the default dependency historical observations can be used in a direct way. Furthermore the model links naturally with portfolio credit risk models like CreditMetrics. The dynamics of the model are realistic, too: Credit spread changes at credit crises happen in an endogenous and intuitive way, and a realistic time-distribution of the times of default is achieved. Existing portfolio credit-risk models of the intensity-type were only partially able to achieve these goals.

These features of the model make it ideally suited to the analysis of the pricing and hedging of basket default swaps and CDOs and counterparty risk in credit derivatives.

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