A GENERALIZED FRAMEWORK FOR CREDIT RISK PORTFOLIO MODELS

H. UGUR KOYLUOGLU
Oliver, Wyman & Company
666 Fifth Avenue
New York, New York 10103
e-mail: ukoyluoglu@owc.com

ANDREW HICKMAN
CSFP Capital, Inc.*
Eleven Madison Avenue
New York, New York 10010
e-mail: andrew.hickman@csfp.co.uk

version: 14 September, 1998

ABSTRACT

Sophisticated new credit risk portfolio modeling techniques have been developed recently, holding the potential for substantial reform of credit risk management techniques and regulatory capital guidelines. This paper examines three such credit risk portfolio models, placing them within a single general framework and demonstrating that they are little different in either theory or results, provided that input parameters are harmonized. This result, that a strong consensus has emerged in the approach to modeling default risk, has significant implications for the acceptance of these new techniques amongst both end-users and regulators.

Helpful comments and discussions were provided by Andrew Cross, Frank Diebold, Tom Garside, Marc Intrater, Andrew Kuritzkes, Hashem Pesaran, Til Schuermann, James Wiener, and Tom Wilde. All errors are the responsibility of the authors alone.

An abridged version of this paper is forthcoming in *Risk* magazine.

*CSFP Capital, Inc. is an indirect subsidiary of Credit Suisse Financial Products

printed: 16 October, 1999

In the past few years, major advances in credit risk analytics have led to the proliferation of a new breed of sophisticated credit portfolio risk models. A number of models have been developed, including both proprietary applications developed for internal use by leading-edge financial institutions, and third party applications intended for sale or distribution as software. Several have received a great deal of public attention, including J.P. Morgan's *CreditMetrics*/*CreditManager*, Credit Suisse Financial Products' *CreditRisk+*, McKinsey & Company's *CreditPortfolioView*, and KMV's *PortfolioManager*. These new models allow the user to comprehensively measure and quantify credit risk at both the portfolio and contributory level, which was not possible previously. As such, they have the potential to cause profound changes to the lending business, accelerating the shift to active credit portfolio management[1], and eventually leading to an "internal models" reform of regulatory credit risk capital guidelines[2].
But before these models can deliver on their promise, they must earn the acceptance of credit portfolio managers and regulators. To these practitioners, this seemingly disparate collection of new approaches may be confusing, or may appear as a warning sign of an early developmental stage in the technology. While these misgivings are understandable, this paper will demonstrate that these new models in fact represent a remarkable consensus in the underlying framework, differing primarily in calculation procedures and parameters rather than financial intuition.

This paper explores both the common ground and the differences amongst the new credit risk portfolio models, focusing on three representative models:

1. "Merton-based" – e.g. *CreditMetrics* and *PortfolioManager*[3]
2. "Econometric" – e.g. *CreditPortfolioView*
3. "Actuarial" – e.g. *CreditRisk+*

Section I provides a quick review of the basic structure of each of the models. Section II describes the common underlying framework of these models and how the models and their assumptions can be related to the generalized framework. Section III draws linkages between the equivalent parameters in each model. Section IV assesses the impact of their differing assumptions using illustrative examples. Section V presents conclusions.

Note that this paper examines only the default component of portfolio credit risk. Some of the models incorporate credit spread (or ratings migration) risk, while others advocate a separate model. In this aspect of credit risk there is less consensus in modeling techniques, and the differences need to be explored and resolved in future research. The reader should strictly interpret "credit risk" to mean "default risk" throughout this paper.
Additionally, for comparability, the models have been restricted to the case of a single-period horizon, fixed recovery rate, and fixed exposures. This should not significantly affect the conclusions, as default dominates the contribution from these other effects, and the mechanisms to incorporate these effects are not too deeply entangled with the models of default.

I. Overview Models

The following sections briefly describe the calculation procedures of each of the models, modified to the two-state (default or not), single-period, fixed recovery, and fixed exposure restrictions.

CreditMetrics

CreditMetrics is a Merton-based model, relying on Merton's model of a firm's capital structure[4]: a firm defaults when its asset value falls below its liabilities. A borrower's default probability then depends on the amount by which assets exceed liabilities, and the volatility of those assets. If changes in asset value are normally distributed, the default probability can be expressed as the probability of a standard normal variable falling below some critical value. The first step in this model is then to calculate critical values corresponding to each borrower's default probability (mapped from the borrower's credit rating). Joint default events amongst borrowers in the portfolio are related to the extent that the borrowers' changes in

---

[1] for example, see Kuritzkes (1998).
[2] see ISDA (1998).
[3] the discussions which follow will focus on *CreditMetrics* as the example, but will also apply reasonably well to *PortfolioManager*.
[4] see Merton (1974), Kealhoffer (1995), and Gupton, Finger, and Bhatia (1997).

asset value are correlated (input in the form of a pairwise correlation matrix determined according to country and industry groupings). The portfolio loss distribution is calculated by Monte Carlo simulation, as follows:

1. Draw random correlated standard normal variables representing the change in asset value for each borrower.
2. Compare this standardized change in asset value to the pre-calculated critical value to determine which borrowers default.
3. Sum the losses resulting from each borrower default to arrive at a total portfolio loss.
4. Repeat thousands of times to build a distribution of portfolio losses.

CreditPortfolioView

CreditPortfolioView posits an empirically derived relationship which drives each borrower's (or "segment" of borrowers') default rate $p_{i,t}$ according to a normally distributed "index" of macroeconomic factors for that borrower[5]. The macroeconomic index $y_{i,t}$ is expressed as a weighted sum of macroeconomic variables, $x_{k\,t}$, each of which is normally distributed and can have lagged dependency.

$x_{k,t} = a_{k,0} + a_{k,1}x_{k,t-1} + a_{k,2}x_{k,t-2} + \ldots + e_{k,t}$, and

$y_{i,t} = b_{i,0} + b_{i,1}x_{1,t} + b_{i,2}x_{2,t} + \ldots + u_{i,t}$,

where the $e_{k\,t}$ and $u_{i,t}$ are normally distributed random innovations.

The index is transformed to a default probability by the Logit function:

$$p_{i,t} = \frac{1}{1+e^{y_{i\,t}}}.$$

The factor loadings $b_{i,k}$ for the index are determined by the empirical relationship between sub-portfolio default rates and explanatory macroeconomic variables, using logistic regression. The coefficients $a_{k,j}$ to the macro-economic variables can be determined by an appropriate econometric model[6].

The portfolio loss distribution is calculated by Monte Carlo simulation, as follows:

1. Draw random innovations to each macroeconomic variable and index value according to their covariance structure.
2. Calculate:
   a) macroeconomic variables outcomes according to their lagged past values and random innovations;
   b) index values according to the macroeconomic values and the index random innovations; and
   c) resulting default probabilities.
3. Calculate the distribution of default outcomes for this iteration by successively convoluting each obligor's (two-state) distribution of outcomes.
4. Repeat thousands of times to build a distribution of portfolio losses.

CreditRisk+

CreditRisk+ makes use of mathematical techniques common in loss distribution modeling in the insurance industry[7]. Joint-default behavior of borrowers is incorporated by treating the default rate as a random variable common to multiple borrowers. Borrowers are allocated amongst "sectors" each of which has a mean default rate and a default rate volatility. The default rate volatility is the standard deviation which would be observed on an infinitely diversified homogeneous portfolio of borrowers in the sector. The default rate x_k for the kth sector is assumed to follow a Gamma distribution, with parameters a_k and b_k set to yield a given mean default rate m_k and volatility of default rate s_k:

$x_k \sim \Gamma[a_k, b_k]$,

[5] see Wilson (1997).
[6] Wilson (1997) suggests several possibilities.
[7] see Credit Suisse Financial Products (1997).

where $\alpha_k = \frac{{\mu_k}^2}{{\sigma_k}^2}$, and $\beta_k = \frac{{\sigma_k}^2}{\mu_k}$.

In the single-sector case, a borrower's default rate is scaled to this Gamma-distributed sector default rate. For multiple sectors, a borrower's default rate is scaled according to the weighted average of sector default rates:

$$p\big|_X = \overline{p}\sum_k w_k \frac{x_k}{m_k},$$

where $\overline{p}$ is the unconditional default rate, and w_k is the weight in sector k, $\sum_k w_k = 1$.

For a homogeneous sub-portfolio of borrowers (same sector and same exposure size) with independent joint-default behavior, *CreditRisk+* assumes that the default distribution follows the Poisson distribution. Since borrowers' joint-default behaviors are independent conditional on fixed default rates, the unconditional default distribution for the homogeneous sub-portfolio can be obtained by "averaging" Poisson conditional default distributions according to default rates from the Gamma distribution – statistically, the convolution of the Poisson distribution with the Gamma distribution. This convolution leads to an analytic expression for the resulting unconditional distribution of portfolio losses.

As presented so far, these models appear to be quite dissimilar. *CreditMetrics* is a bottom-up model (each borrower's default is modeled individually) with a microeconomic causal model of default (the Merton model). *CreditPortfolioView* is a bottom-up model based on a macroeconomic causal model of sub-portfolio default rates. *CreditRisk+* is an almost entirely top-down model of sub-portfolio default rates, making no assumptions with regard to causality. Despite these apparent differences, each of these models can be demonstrated to fit within a single generalized underlying framework, presented in the following section.

II. Generalized Underlying Framework for Credit Portfolio Modeling

The generalized credit portfolio model consists of three main components to calculate the portfolio loss distribution:

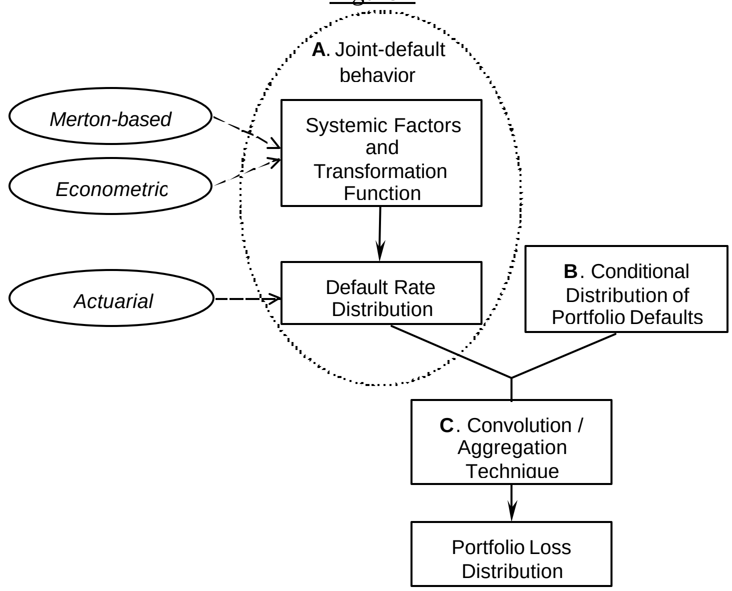
A. *Joint-default behaviour* – Default rates vary over time, intuitively as a result of varying economic conditions; when conditions are favorable, fewer borrowers default, and vice versa. A conditional default rate is generated for each borrower in each "state of the world" for the relevant economic conditions. The degree of "concentration" or "correlation" in the portfolio is reflected by the extent to which the borrowers' conditional default rates vary together in different "states of the world".

B. *Conditional distribution of portfolio default rate* – for each "state of the world" and its corresponding set of borrowers' conditional default rates, the conditional distribution of a homogeneous sub-portfolio default rate can be calculated as if individual borrower defaults are independent, as all of the joint-default behavior has been accounted for in generating conditional default rates.

C. *Convolution / Aggregation* – the unconditional distribution of portfolio defaults is obtained by aggregating homogeneous sub-portfolio's conditional distribution of default rate in each "state of the world", and then by simply averaging across the conditional distributions of portfolio defaults for different states of the world, weighted by the probability of a given state.

This generalized framework (see Figure 1) allows a structured comparison of the models. The following sections explain how each of the three models approaches these components, whether explicitly or implicitly.

Figure 1



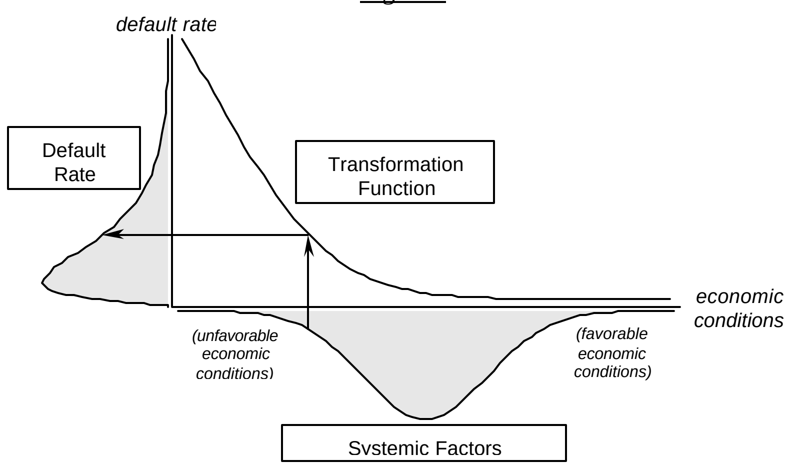
II. A. Conditional Default Rates and Probability Distribution of Default Rate

All three models explicitly or implicitly relate default rates to variables describing the relevant economic conditions ("systemic factors"). This relationship can be expressed as a transformation function, the "conditional default rate" function (see Figure 2). This function is explicitly assigned in the econometric model, and can be derived in closed form for both the Merton-based and actuarial models.

The systemic factors are random, and are usually assumed to be normally distributed. Since the conditional default rate is a function of these random systemic factors, the default rate will also be random. The default rate distribution is an explicit assumption in the actuarial model, and an implicit assumption in the Merton-based and econometric models. For purposes of comparison, the default rate distributions implied by the latter two models can be derived easily.

The relationship between the systemic factor distribution and the default rate distribution is represented graphically in Figure 2.

Figure 2



Merton model

Since the Merton model neither assigns the transformation function, nor assumes a probability distribution for default rates explicitly, these relationships must be derived.

As explained in section I, a borrower's default is motivated by a normally distributed change in (standardized) asset value ΔA_i, which is correlated with the change in asset value of other borrowers in the portfolio. This change in asset value can be decomposed into a set of normally distributed orthogonal systemic factors x_k, and a normally distributed idiosyncratic component ε_i:

$$\Delta A_i = b_{i,1}x_1 + b_{i,2}x_2 + \ldots + \sqrt{1-\sum_k b_{i,k}{}^2}\, e_i ,$$

where $b_{i,k}$ are the factor-loadings, $x_k, e_i \sim iid\ N[0,1]$, and consequently $\Delta A_i \sim N[0,1]$.

If the values of the systemic factors are known, then the standardized change in asset value will be normally distributed with a mean given by the factor loadings and factor values, and a standard deviation given by the weight of the idiosyncratic factor. These systemic factors and factor loadings can be selected in such a way as to exactly replicate the pairwise asset correlation structure[8] (in general this will require N-1 factors for N borrowers, fewer if the pairwise asset correlation structure was created from a smaller set of industry and country groupings). Alternatively, a smaller set of systemic factors might be selected.

According to the Merton model, default occurs when $\Delta A_i \le c$, where the "critical value" *c* is calibrated to provide the correct unconditional default probability $\overline{p}$; that is $\Phi(c) = \overline{p}$, where $\Phi[x]$ is the cumulative density function of the normal distribution. The default rate, conditioned on the values of systemic factors, can then be expressed as[9]

$$p_i\big|_X = \Phi\left[\frac{c-\sum_k b_{i,k}x_k}{\sqrt{1-\sum_k b_{i,k}{}^2}}\right].$$

For the single borrower or homogeneous portfolio case, the systemic factors can be summarized by a single variable *m*, reducing the transformation function to

$$p\big|_m = \Phi\left[\frac{c-\sqrt{r}\,m}{\sqrt{1-r}}\right],$$

where $m \sim N[0,1]$ and $r = \sum_k b_k{}^2$ is the asset correlation in the homogeneous portfolio.

Since the cumulative normal function is bounded [0,1] and concave in the relevant region, the resulting default rate distribution is bounded in [0,1] and skewed to the right as in Figure 2.

The probability density function for the default rate $f(p)$ can be derived explicitly, as it is related to the probability density function of systemic factors $j(m)$ by the following:

$$f(p) = j\left(m(p)\right)\left|\frac{dm}{dp}\right| = \frac{j\left(m(p)\right)}{\left|\frac{dp}{dm}\right|}.$$

Applied to the Merton-based model's transformation function and normally-distributed systemic factors, this yields

$$f(p) = \frac{\sqrt{1-r}\ j\left(\dfrac{c-\sqrt{1-r}\ \Phi^{-1}(p)}{\sqrt{r}}\right)}{\sqrt{r}\ j\left(\Phi^{-1}(p)\right)},$$

where $j(z)$ is the standardized normal density function.

[8] The correlations might be derived by Cholesky decomposition or Principal Components Analysis, for example.

[9] Vasicek (1987) develops this representation of the Merton model for a single factor.

Econometric model

As in section I, the econometric model explicitly defines the borrower's default probability as a function of an index value:

$$p_{i,t} = \frac{1}{1+e^{y_{i,t}}}.$$

This index value in turn depends on macroeconomic variables. The index and macroeconomic variable models can be combined to a single equation for the index:

$$y_{i,t} = \left(b_{i,0} + \sum_k b_{i,k} \left(a_{k,0} + \sum_j a_{k,j} x_{k,t-j} \right) \right) + \sum_k b_{i,k} e_{k,t} + u_{i,t}.$$

The index thus consists of a constant term (itself composed of a constant and the weighted lagged values of macroeconomic variables), and random terms representing normally distributed systemic and index-specific innovations. For the single borrower or homogeneous portfolio case, the normally distributed systemic and index-specific innovations can be summarized by a single normally distributed variable m, so that

$$y_{i,t} = U_i + V_i m_i \ ,$$

where

$$U_i = b_{i,0} + \sum_k b_{i,k} \left(a_{k,0} + \sum_j a_{k,j} x_{k,t-j} \right),$$

$$V_i = \sqrt{\operatorname{var}(u_{i,t}) + \sum_k \left(2b_{i,k} \operatorname{cov}(u_{i,t}, e_{k,t}) + b_{i,k}{}^2 \operatorname{var}(e_{k,t}) + \sum_{m \neq k} b_{i,k} b_{i,m} \operatorname{cov}(e_{k,t}, e_{m,t}) \right)},$$

and $m \sim N[0,1]$.

The conditional default rate function can then be expressed as

$$p\,|_m = \frac{1}{1+e^{U+Vm}}.$$

Since the Logit function is bounded [0,1] and concave, the resulting distribution is bounded [0,1] and skewed to the right as in Figure 2.

Deriving the implied probability density function for the default rate $f(p)$ proceeds just as in the Merton-based model, yielding

$$f(p) = \frac{1}{V p\,(1-p)}\, j\left(\frac{\ln\left(\frac{1-p}{p}\right) - U}{V} \right).$$

Actuarial model

The actuarial model assumes that the default rate distribution $f(p; m, s)$ follows a Gamma distribution, or in the multi-sector case, a weighted-average of Gamma distributions[10]. The Gamma distribution is bounded at left by zero but has infinite positive support. It is possible to derive the actuarial model's implied transformation function such that when applied to a normally distributed systemic factor, it results in a Gamma distribution for the default rate. The transformation function consists of all points (c, x) which satisfy:

$$\int_0^x \Gamma(p; a, b)\; dp = \int_c^\infty j\,(m)\; dm.$$

[10] Only the single-sector case is considered in the discussion which follows. A weighted-average of Gamma distributions will not in general be a Gamma distribution itself. The equations throughout the remainder of this paper can be modified to the multi-sector case with moderate additional complexity. However, all else equal, this should not make much difference.

Hence, the transformation function is given by:

$$p|_m = \Psi^{-1}(1-\Phi(m);\ a, b),$$

where $a = \frac{\bar{p}^2}{s^2}$, $b = \frac{s^2}{\bar{p}}$, and $\Psi(z;a,b)$ is the cumulative density function of the Gamma distribution $\Gamma(z;a,b)$.

Note that all three models are related as described to normally distributed systemic factors, but the normal distribution is not a critical assumption. The default rate distributions for the Merton-based and econometric models, and the implied conditional default rate function in the actuarial model, can easily be calculated for an arbitrary systemic factor distribution. While non-normality may affect the results of model comparisons, it does not in principle render them incomparable.

The three conditional default rate functions and default rate distributions are compared in section IV.

II. B. Conditional Distribution of Portfolio Default Rate

Under the condition that all loans are independent, given fixed or conditional default rates, a homogeneous sub-portfolio's distribution of defaults follows the Binomial distribution $B(k;n,p)$, which provides the probability that k defaults will occur in a portfolio of n borrowers given that each has probability of default p:

$$B(k;n,p) = \frac{n!}{k!\,(n-k)!} p^k\,(1-p)^{n-k}.$$

CreditMetrics implicitly uses the Binomial distribution by calculating the change in asset value for each borrower and testing for default – exactly equivalent to the Binomial case of two states with a given probability. *CreditPortfolioView* explicitly uses the Binomial distribution by iteratively convoluting the individual obligor distributions, each of which is Binomial. *CreditRisk+* approximates the Binomial with the Poisson distribution $P(k;pN)$, which provides the probability that k defaults will occur in a portfolio of n borrowers given that each occurrence has a rate of intensity per unit time p:

$$P(k;pN) = \frac{(pn)^k}{k!} e^{-pn}.$$

The Binomial and Poisson distributions are quite similar. Indeed, the Poisson distribution is easily shown to be the limiting distribution for the Binomial distribution, as the number of borrowers becomes large and the default probability becomes small[11]. The Poisson distribution does allow for the possibility of multiple defaults for a single borrower, but, for reasonably small default rates, the probability of multiple defaults is negligible. For portfolios which are heavily concentrated in a few names, or do not have a large enough number of borrowers, the Binomial and Poisson distributions may show differences; but in these cases, credit risk models may not be relevant anyway, as the question reduces to whether or not particular borrowers default. The difference between Binomial and Poisson should not be significant for reasonable portfolios[12].

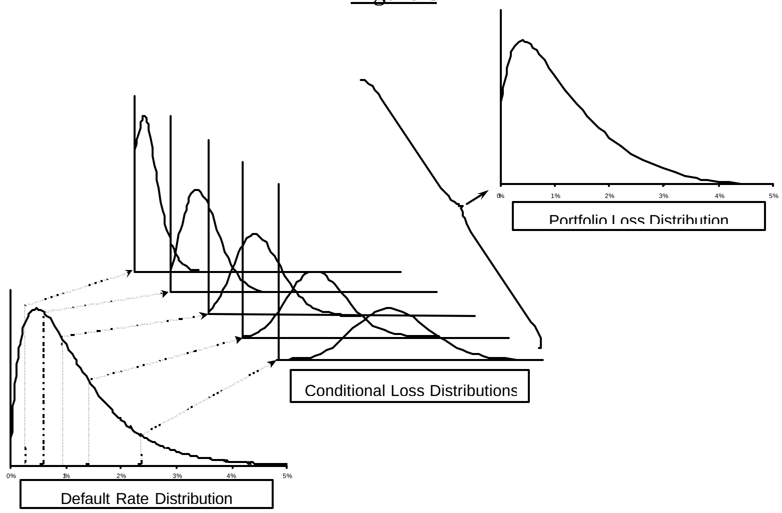
II. C. Aggregation

The unconditional probability distribution of portfolio defaults is obtained by aggregating the conditional distributions of portfolio defaults under all possible "states of the world" for relevant economic conditions. This is simply calculated by averaging across the conditional distributions of portfolio defaults for different "states of the world", weighted by the probability of a given state, as depicted in Figure 3. Mathematically, this is expressed as a convolution integral.

[11] see Freund (1992).

[12] Stuart & Ord (1994) provides expressions for the maximum difference between the Binomial and Poisson distributions.

Figure 3



The Merton-based and econometric models are conditioned on normally distributed systemic factors, and the independent loans are Binomially distributed. Hence, the convolution integral for a homogeneous sub-portfolio with a single systemic factor is expressed as

$$\text{Prob}(k \text{ defaults in a sub-portfolio of } n \text{ borrowers}) = \int_{-\infty}^{\infty} B(k;n,p|_m)\ j(m)\ dm\ .$$

The actuarial model is conditioned on the random default rate, and the independent loans are assumed to be Poisson distributed. Therefore, the convolution integral for a homogeneous sub-portfolio is given by:

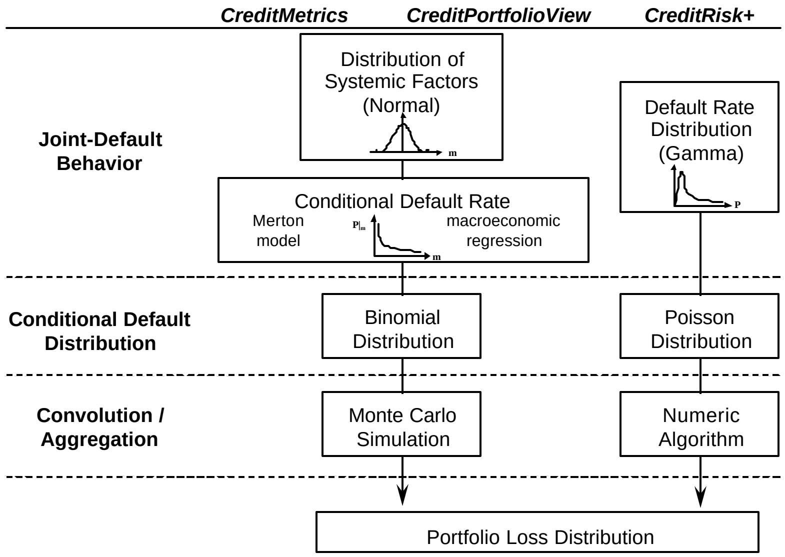
$$\text{Prob}(k \text{ defaults in a sub-portfolio of } n \text{ borrowers}) = \int_{0}^{\infty} P(k;np)\ \Gamma(p;a,b)\ dp\ .$$

These integrals are easily evaluated; in particular, the convolution of the Poisson distribution and Gamma distribution yields a closed-form distribution, the Negative Binomial Distribution. It is the differences between sub-portfolios – differing exposure size or default probabilities, or multiple systemic factors, complex correlation structure, etc. – that create difficulty in aggregation. *CreditMetrics* performs a Monte Carlo simulation of both the systemic factors and the individual borrower default outcomes. Monte Carlo simulation introduces sampling error which can be mitigated by increasing the number of simulations. *CreditPortfolioView* also uses Monte Carlo simulation of systemic factors (and hence default probabilities), and then evaluates the conditional portfolio distribution with a convolution algorithm. This algorithm tallies the portfolio distribution in a "grid", causing an approximation error which can be mitigated by increasing the number of grid points. *CreditRisk+* evaluates the convolutions with a numeric algorithm. This numeric algorithm introduces an approximation error in the "banding" of exposures[13], which can be mitigated by decreasing the "unit size". In all three cases, the procedures are theoretically exact in the limit.

Figure 4 depicts the models as they are redefined in relation to the generalized framework, highlighting the specific components of each.

[13] See CSFP (1997) for analysis of the magnitude of this approximation error.

Figure 4



III. Harmonization of joint-default parameters

The preceding discussion of the models' workings highlights that all three models critically depend on two elements of information: unconditional default probability and joint-default behavior. While unconditional default probability appears in a relatively consistent, straightforward manner in each model, joint-default behavior appears in a variety of forms. The Merton-based model uses pairwise asset correlations; the actuarial model uses sector default rate volatilities and borrower sector weightings; and the econometric model calculates regression coefficients to macroeconomic factors which incorporate correlations amongst the factors. Although these parameters are very different in nature, they all are related to each other and should contain equivalent information to characterize joint-default behavior.

Coefficients and correlations

As described in section II.A, joint-default behavior is represented in Merton-based models as a pairwise asset correlation matrix, or equivalently as a set of asset factor-loadings for each borrower:

$$\Delta A_i = b_{i,1}x_1 + b_{i,2}x_2 + \ldots + \sqrt{1-\sum_k b_{i,k}{}^2}\, e_i \, .$$

The systemic factors are defined to be orthonormal, so that

$$correlation[\Delta A_i, \Delta A_j] = \frac{E[\Delta A_i \Delta A_j] - E[\Delta A_i]E[\Delta A_j]}{\sqrt{(E[\Delta A_i{}^2] - E[\Delta A_i]^2)(E[\Delta A_j{}^2] - E[\Delta A_j]^2)}} = b_{i,1}b_{j,1} + b_{i,2}b_{j,2} + \ldots$$

Using this relationship, a pairwise correlation matrix is easily calculated given asset factor-loadings, and factor-loadings can be derived from a pairwise correlation matrix (though the factors will not be specified). Hence asset correlation will be related to other joint-default behavior parameters, as it is usefully a single statistic for any pair of borrowers; asset factor-loadings will first need to be translated to asset correlation in order to make use of the relationships which will be derived.

The econometric model's logistic regression coefficients, which define the relationship of the default rate "index" to macroeconomic variables, bear strong resemblance to the asset factor loadings of the Merton-based model. In fact, an "index correlation" is easily defined in a similar fashion. First, as in section II.1, the index is re-stated in terms of index coefficients and random innovations as well as macroeconomic variable coefficients and random innovations:

$$y_{i,t} = \left(b_{i,0} + \sum_k b_{i,k} \left(a_{k,0} + \sum_j a_{k,j} x_{k\,t-j} \right) \right) + \sum_k b_{i,k} e_{k,t} + u_{i,t} \, .$$

Then,

$$correlation[y_i, y_j] = \frac{E[y_{i,t} y_{j,t}] - E[y_{i,t}]E[y_{j,t}]}{\sqrt{\mathrm{var}(y_{i,t})\,\mathrm{var}(y_{j,t})}}$$

$$= \frac{\mathrm{cov}(u_{i,t}, u_{j,t}) + \sum_k \left(b_{i,k}\,\mathrm{cov}(u_{j,t}, e_{k,t}) + b_{j,k}\,\mathrm{cov}(u_{i,t}, e_{k,t}) + \sum_m b_{i,k} b_{j,m}\,\mathrm{cov}(e_{k,t}, e_{m,t}) \right)}{V_i V_j}$$

where $V_i = \sqrt{\mathrm{var}(u_{i,t}) + \sum_k \left(2b_{i,k}\,\mathrm{cov}(u_{i,t}, e_{k,t}) + b_{i,k}{}^2\,\mathrm{var}(e_{k,t}) + \sum_{m \neq k} b_{i,k} b_{i,m}\,\mathrm{cov}(e_{k,t}, e_{m,t}) \right)}$.

Asset correlation and "index correlation" are very similar in concept, but will provide slightly different results to the extent that their respective conditional default rate functions are different (see section IV). For the purposes of translation to other forms of joint-default parameters, the degree of similarity should be sufficient; accordingly, only asset correlation will be considered in the discussion which follows. The interested reader is encouraged to work out the relationships for index correlations as distinct from asset correlations using the same techniques.

Default rate volatility

Default rate volatility s is calculated by the standard formula for variance:

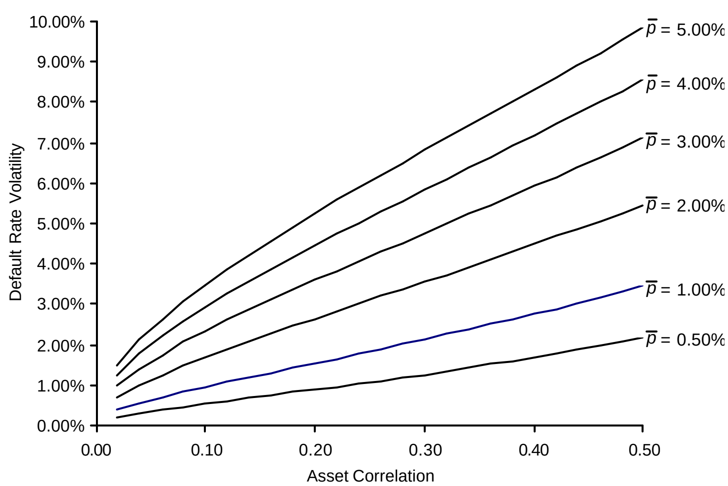
$$s^2 = \int_0^\infty (p - \overline{p})^2 f(p)\,dp\ .$$

In particular, for the Merton model, the default rate volatility can be expressed as a function of r and $\overline{p}$:

$$s^2 = \int_{-\infty}^{\infty} (p|_m - \overline{p})^2 j(m)\,dm = \int_{-\infty}^{\infty} \left(\Phi\left[\frac{\Phi^{-1}[\overline{p}] - \sqrt{r}\,m}{\sqrt{1-r}}\right] - \overline{p}\right)^2 j(m)\,dm\ .$$

Figure 5 shows the resulting relationship between asset correlation and default rate volatility.

Figure 5



Default correlation

Some applications, and indeed some models, take a Markowitz variance-covariance view to credit risk portfolio modeling. Each borrower has a variance of default given by the variance for a Bernoulli variable: $\mathrm{var}(default_i) = \overline{p}_i(1 - \overline{p}_i)$.

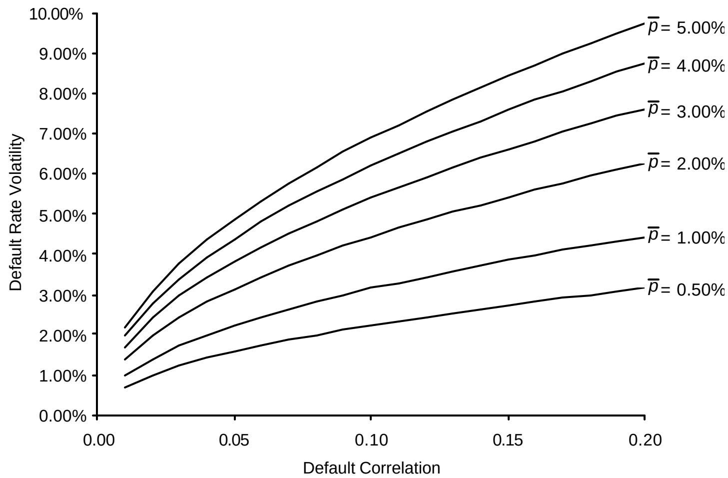
A pairwise default correlation matrix is specified, and the portfolio variance (due to defaults) can be calculated by the standard $w \cdot \Sigma \cdot w^T$ operation (w is the position vector – in this case containing each borrower's exposure net of recovery – and Σ is the covariance matrix).

For a homogeneous portfolio with a large number of names, the portfolio variance approaches

$$s^{2} = \overline{p}(1-\overline{p})\ r_{default}\ .$$

As the infinite homogeneous portfolio's default rate volatility is precisely the parameter in the actuarial model, this provides the relationship between default correlation and default rate volatility for two borrowers with the same unconditional default rate[14]. This relationship is illustrated for various levels of unconditional default rate in Figure 6.

Figure 6



In general, default correlation is calculated by the standard formula

$$r_{default}(i,j) = \frac{E[\text{default}_{\text{i}} \text{ and default}_{\text{j}}] - E[\text{default}_{\text{i}}]E[\text{default}_{\text{j}}]}{\sqrt{\text{var}(\text{default}_{\text{i}})\,\text{var}(\text{default}_{\text{j}})}} = \frac{E[\text{default}_{\text{i}} \text{ and default}_{\text{j}}] - \overline{p}_i \overline{p}_j}{\sqrt{(\overline{p}_i(1-\overline{p}_i))(\overline{p}_j(1-\overline{p}_j))}}\ ,$$

where

$$E[\text{default}_{\text{i}} \text{ and default}_{\text{j}}] = \int_0^\infty \int_0^\infty p_i\ p_j\ f(p_i, p_j)\ dp_i\ dp_j\ ,$$

or, if the default rate can be stated in terms of a systemic factor,

$$E[\text{default}_{\text{i}} \text{ and default}_{\text{j}}] = \int_{-\infty}^{\infty} p_i\big|_m\ p_j\big|_m\ j(m)\ dm\ .$$

Reverting to the causal model of default in the Merton model, a simplified expression can be derived for default correlation in the Merton model, as joint-default occurs only when the two borrowers' (standardized) asset values fall below their respective critical values:

$$E[\text{default}_{\text{i}} \text{ and default}_{\text{j}}] = \int_{-\infty}^{c_i} \int_{-\infty}^{c_j} f(\Delta A_i, \Delta A_j; r)\ d\Delta A_i\ d\Delta A_j\ .$$

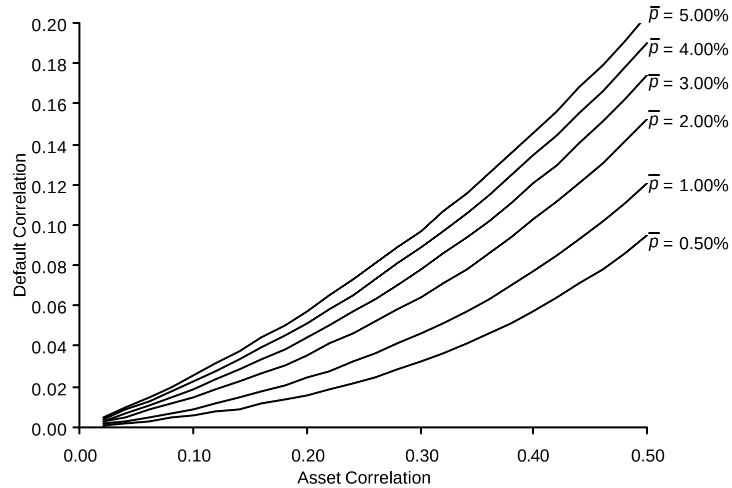
The joint distribution of two borrowers' changes in asset values is the bivariate normal distribution:

$$f(\Delta A_i, \Delta A_j; r) = \frac{1}{2p\sqrt{1-r^2}} e^{\frac{\Delta A_i^2 - 2r\Delta A_i \Delta A_j + \Delta A_j^2}{2(1-r^2)}}\ .$$

Thus calculated, the relationship between asset correlation and default correlation is illustrated in Figure 7.

[14] CSFP (1997) provides a general formula for the default correlation between two borrowers with differing default probabilities, given their sector allocations and the sector default rate volatilities.

Figure 7



This calculation can be performed for differing default probabilities as well. Note that it also applies to the multi-sector case, as for any pair of borrowers, the systemic factors can be summarized to a single factor; i.e. two borrowers always have just one asset correlation no matter how many factors there are.

These mappings in Figures 5, 6, and 7 relate the joint-default parameters in all three models as well as the "covariance" model. Such mappings are especially useful in parameter estimation. For instance, in the absence of equity prices, default rate volatilities can be used to estimate implied asset correlations. The mappings also allow parameter estimates to be "triangulated" by multiple methods, to the extent that model differences are not significant.
Note, however, that these illustrated mappings assume the homogeneous portfolio case ($\overline{p}_i = \overline{p}_j = \overline{p}$); if the default rate differs between borrowers, the more general translation formulae will be required.

IV. Differences in the Default Rate Distribution
The discussion in section II demonstrates that substantial model differences could arise only from the differing treatment of joint-default behavior – the conditional default distributions are effectively the same for the relevant cases, and the convolution and aggregation techniques are theoretically exact in the limit. Section III has provided the means to compare the joint-default behavior on an apples-to-apples basis. This comparison will be illustrated for a homogeneous portfolio with an unconditional default rate $\overline{p}$ of 116bp and a standard deviation of default rate s equal to 90bp[15]. Since each model produces a two-parameter default rate distribution, the mean and standard deviation are sufficient statistics to define the parameters for any of the models. Before the comparisons can be performed, the relevant parameters for each model must be derived such that the models match the unconditional default rate and standard deviation of default rate.

Merton-based model
The merton model requires two parameters: the critical value c, and the asset correlation r . The crtical value is defined in terms of the unconditional default probability:

$$c = \Phi^{-1}(\overline{p})$$

As in section III, the default rate volatility is calculated by

$$s^{2} = \int_{-\infty}^{\infty} (p|_m - \overline{p})^2 \, j(m)\, dm = \int_{-\infty}^{\infty} \left(\Phi\left[\frac{c - \sqrt{r}\, m}{\sqrt{1-r}} \right] - \overline{p} \right)^2 j(m)\, dm \,,$$

which depends only on c and r .

---

[15] These parameters were selected to match Moody's "All Corporates" default experience for 1970-1995, as reported in Carty & Lieberman (1996).

In order to yield $\bar{p}$ = 116bp and s = 90bp, the parameters for the Merton-based model are c = -2.27 and r = 0.073.

Econometric model
The econometric model requires a variety of constants and coefficients which, as in section II.A, can be summarized by two parameters U and V. The two parameters are found by solving the following system of equations:

$$\bar{p} = \int_{-\infty}^{\infty} \frac{1}{1+e^{U+Vm}}\, j(m)\,dm \text{, and}$$

$$s^2 = \int_{-\infty}^{\infty} \left(\frac{1}{1+e^{U+Vm}} - \bar{p} \right)^2 j(m)\,dm \;.$$

In order to yield $\bar{p}$ = 116bp and s = 90bp, the parameters for the econometric model are U = 4.684 and V = 0.699.

Actuarial model
The parameters required by the actuarial model are defined directly in terms of the unconditional default rate and default rate volatility:

$$a = \frac{\bar{p}^2}{s^2} \text{, and } b = \frac{s^2}{\bar{p}} \;.$$

In order to yield $\bar{p}$ = 116bp and s = 90bp, the actuarial model's Gamma distribution parameters are a = 1.661 and b = 0.0070.

Figure 8 compares the conditional default rate functions. In this example, the models are virtually indistinguishable when the systemic factor is greater than negative two standard deviations, which accounts for almost 98% of the probability mass. For extremely unfavorable economic conditions (systemic factor less than negative two standard deviations), the econometric model predicts a somewhat higher default rate, and the actuarial model predicts a somewhat lower default rate.

Figure 8

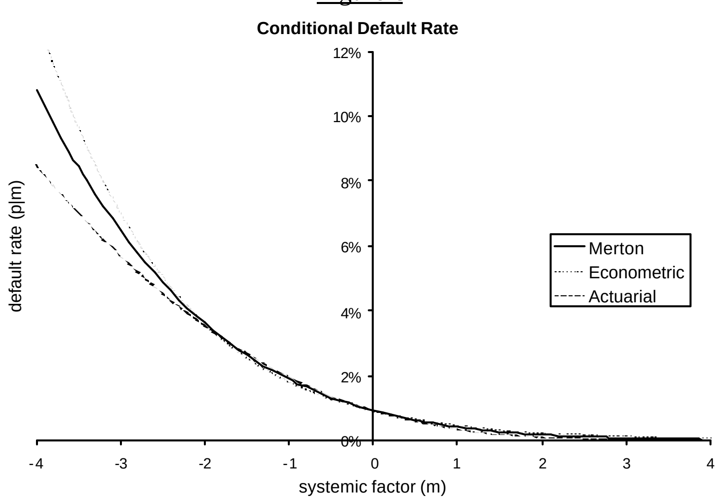


Figure 9 compares the default rate distributions, which naturally show a similar result. These models imply very similar probability density functions for the default rate, with only minor discrepancies at the tail. Figure 10 compares the right tails of these default rate distributions. Note that both of these figures show default rate distributions, not portfolio loss distributions; the portfolio loss distribution approaches the default rate distribution as the number of borrowers becomes very large.

Figure 9

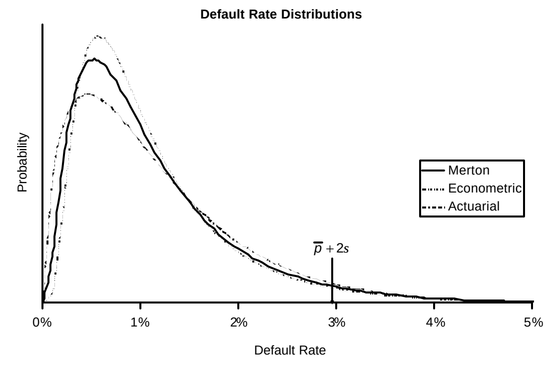
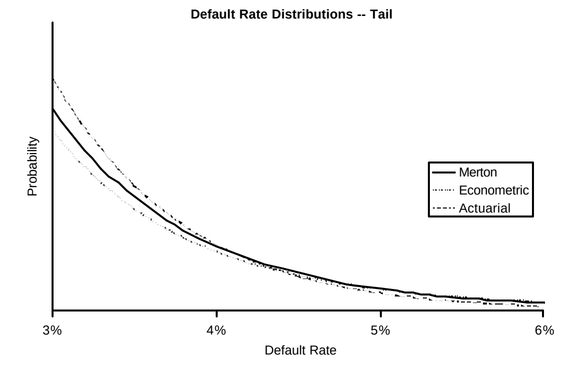


Figure 10



The degree of agreement in the tails of these distributions can be assessed with the following statistic:

$$\Xi_z(f,g) = 1 - \frac{\int_z^\infty \left| f(x) - g(x) \right| dx}{\int_z^\infty f(x)\, dx + \int_z^\infty g(x)\, dx},$$

where $f(x)$ and $g(x)$ are probability density functions and z defines the lower bounds of the "tail". This statistic measures the amount of the probability distributions' mass which overlaps in the relevant region, normalized to the total probability mass of the two distributions in that region. The statistic will be bounded [0,1], where zero represents two distributions with no overlapping probability mass, and one represents exact agreement between the distributions. The "tail" has been defined arbitrarily as the area more than two standard deviations above the mean, i.e. $z = \overline{p} + 2s$.

The tail-agreement statistics for the example distributions in Figures 8-10 are given in Table 1.

Table 1

Ξ(Merton vs. Econometric)	94.90%
Ξ(Merton vs. Actuarial)	93.38%
Ξ(Econometric vs. Actuarial)	88.65%

Without a credible alternative distribution (the normal distribution, for example, is obviously inappropriate and would clearly show that the three distributions are relatively very different from normal and very similar to each other[16]), this tail-agreement statistic provides a relative rather than absolute measure. However, it can be used to test the robustness of the similarity to the parameters, as in the Table 2:

[16] Comparison to the normal distribution would yield a tail-agreement statistic of less than 70%, given that the normal distribution has 2.3% probability mass in the tail above two standard deviations whereas the three credit model distributions have around 4.4% - 4.7% probability mass in their tails.

Table 2

$s\ / \bar{p}$

$\bar{p}$	0.50	1.00	2.00	3.00	
0.05%	98.10%	95.94%	91.16%	89.12%	Ξ(Merton vs. Econometric)
	93.53%	88.50%	81.17%	78.99%	Ξ(Merton vs. Actuarial)
	91.71%	84.70%	73.13%	69.04%	Ξ(Econometric vs. Actuarial)
0.10%	97.92%	95.57%	91.15%	88.40%	
	93.75%	88.94%	82.16%	80.40%	
	91.73%	84.78%	73.95%	69.73%	
0.25%	97.61%	94.93%	90.35%	87.86%	
	94.11%	89.73%	83.87%	82.92%	
	91.79%	84.97%	74.83%	71.60%	
0.50%	97.33%	94.41%	89.91%	88.09%	
	94.42%	90.56%	85.71%	85.61%	
	91.88%	85.30%	76.15%	74.28%	
1.00%	97.06%	93.97%	89.89%	88.97%	
	94.93%	91.69%	88.29%	89.38%	
	92.04%	85.93%	78.57%	78.72%	
2.50%	96.62%	93.62%	90.77%	91.33%	
	95.82%	93.94%	93.65%	94.68%	
	92.62%	87.79%	84.77%	87.53%	
5.00%	96.33%	93.85%	92.79%	94.59%	
	97.02%	96.70%	95.24%	81.79%	
	93.55%	90.87%	91.38%	79.96%	
10.00%	96.21%	94.92%	95.79%	N/A*	
	98.55%	95.57%	72.95%		
	95.45%	95.22%	72.41%		

**Not a reasonable combination of parameters – model results become unstable*

The results in Table 2 demonstrate that the similarity of the models holds for a reasonably wide range of parameters. The models begin to diverge at a very high ratio of default rate volatility to default probability, particularly for very low or very high default probabilities (upper right and lower right regions of the table). In the case of very low default probabilities (upper right) the conditional default rate functions of the Merton-based and econometric models become much more concave than that of the actuarial model. In the case of very high default probabilities (lower right), the gamma distribution's lack of upper bound begins to have a significant effect. Accordingly, in very high quality (AA or better) or very low quality (B or worse) portfolios, model selection can make a difference, though there is scant data on which to base such selection. In a portfolio where very high or very low quality sub-portfolios have only moderate weight, these differences should not be significant in aggregation.

This finding, that the models are quite similar across a broad range of reasonable parameters, should be taken with caution, as it hinges on using harmonized parameter values. In practice, the parameter values actually used will vary by estimation technique. The different estimation techniques appropriate to different joint-default parameters may result in estimates which are inconsistent relative to the equivalencies in section III. Even default probabilities may vary considerably depending on the estimation technique, sample, etc. Unsurprisingly, when the parameters do not imply consistent mean and standard deviation of default rate distribution, the result is that the models are significantly different. Such a case is illustrated in Figure 11, where three models are compared on the basis of three hypothetical parameter sets (see Table 3) which are not consistent, though plausibly obtainable for the same portfolio. Table 4 presents the tail-agreement statistics for all possible combinations of model and parameter set.

Figure 11

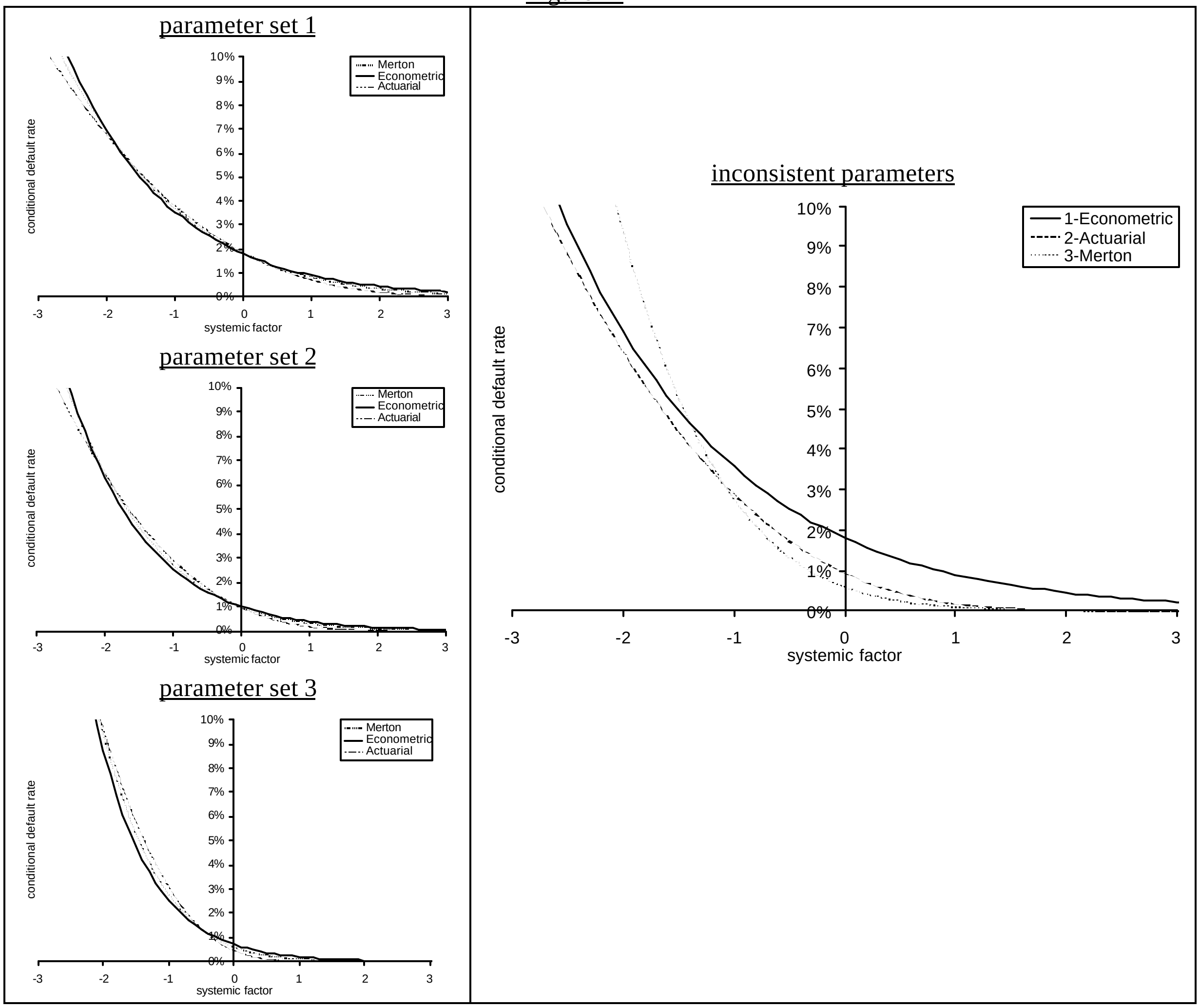


Table 3

Hypothetical inconsistent parameter values:

	$\bar{p}$	s	c	r	a	b	U	V	model for comparison*
1	2.26%	1.70%	-2.00	8.5%	1.767	0.0128	4.00	0.70	Econometric
2	1.52%	1.71%	-2.16	14.4%	0.790	0.0192	4.60	0.95	Actuarial
3	1.54%	2.63%	-2.16	26.2%	0.343	0.0449	4.95	1.30	Merton

**In the inconsistent parameter case, the parameter sets' "models for comparison" were selected arbitrarily. Unshaded cells indicate the parameters appropriate to the selected model.*

Table 4

Tail-agreement statistics** for hypothetical inconsistent parameter sets:

	1_{Merton}	$1_{Econometric}$	$1_{Actuarial}$	2_{Merton}	$2_{Econometric}$	$2_{Actuarial}$	3_{Merton}	$3_{Econometric}$	$3_{Actuarial}$
1_{Merton}	N/A								
$1_{Econometric}$	94.78%	N/A							
$1_{Actuarial}$	94.47%	89.43%	N/A						
2_{Merton}	75.86%	80.74%	71.11%	N/A					
$2_{Econometric}$	69.53%	74.40%	65.02%	93.03%	N/A				
$2_{Actuarial}$	81.63%	**85.00%**	78.05%	91.96%	85.27%	N/A			
3_{Merton}	73.85%	**77.46%**	68.90%	77.74%	72.82%	**77.85%**	N/A		
$3_{Econometric}$	70.87%	75.17%	65.69%	83.36%	80.94%	79.23%	91.28%	N/A	
$3_{Actuarial}$	76.53%	78.77%	72.09%	70.27%	65.55%	73.85%	90.91%	82.49%	N/A

***To allow for comparison between parameter sets, the "tail" was consistently defined as the smallest value of $z = \overline{p} + 2s$, 494bp (parameter set 2)*

Bold figures represent the three possibilities illustrated in the "inconsistent parameters" case in Figure 11.

Table 4 shows that within a consistent parameter set (3x3 boxes on the diagonal), the models are in close agreement, with tail-agreement statistics averaging 91.40% and ranging from 82.49% to 94.78%. Large differences are found between inconsistent parameter sets (3x3 boxes off of the diagonal), with tail-agreement statistics averaging 76.18% and ranging from 65.02% to 85.00%. The differences in parameters, well within the typical range of estimation error, have much greater impact than model differences in this example.

V. Conclusions

On the surface, the credit risk portfolio models studied in this paper seem to be quite different – the approaches range from a bottom-up microeconomic model of an individual borrower's default to a top-down macroeconomic model of the default rate for a sub-portfolio of borrowers. However, deeper examination reveals that the models belong to a single general framework, which identifies three critical points of comparison – the default rate distribution, the conditional default distribution, and the convolution / aggregation technique.

Differences were found to be immaterial in the conditional default distributions and the convolution / aggregation techniques, so that any significant differences between the models must arise from differences in modeling joint-default behavior which manifest in the default rate distribution. Further, when the joint-default parameter values are harmonized to a consistent expression of default rate and default rate volatility, the default rate distributions are sufficiently similar as to cause little meaningful difference across a broad range of reasonable parameter values. Any significant model differences can then be attributed to parameter value estimates which have inconsistent implications for the observable default rate behavior.

Parameter inconsistency is not a trivial issue. A "naïve" comparison of the models, with parameters estimated from different data using different techniques, is quite likely to produce significantly different results for the same portfolio. The conclusions of empirical comparisons of the models will vary according to the degree of difference in parameters[17]. In such comparisons, it is important to understand the proportions of "parameter variance" and "model variance" if different results are produced for the same portfolio. The findings in this paper suggest that "parameter variance" is likely to dominate. Future studies should focus on the magnitude of parameter differences and the sensitivity of results to these differences.

The implications of parameter inconsistency range beyond sample portfolio comparisons. Parameter inconsistency can arise from two sources: (1) estimation error, which could arise from small sample size or other sampling issues, or (2) model mis-specification. While default rate volatility may be immediately observable, even long periods of observation provide small sample size (e.g. 25 years of default rate experience is only 25 data points) and risk change in the underlying behaviors. At the other extreme, asset correlations can be measured with reasonable sample size in much shorter periods, but risk mis-specification in the assumptions with respect to return distributions and default causality in the translation from asset correlations to default rate volatility. Rather than conclude that parameter inconsistency

[17] For example, ISDA (1998) and Roberts and Wiener (1998) compare the results of several models on test portfolios. The former finds that model results are fairly consistent, while the latter finds that the models may produce quite different results for the same portfolio using parameters independently selected for each model.

potentially constitutes irreconcilable differences between the results of these models, this paper concludes that because the models are so closely related, the estimates are complementary and should provide improved accuracy in parameter estimation within the generalized framework as a whole.

The findings of this paper do not in any way suggest that one of the models is necessarily superior to the others, nor do they suggest that users should be indifferent between the models. Rather, these findings suggest that relative "theoretical correctness" need not rank among a user's model selection criteria, which might then consist primarily of practical concerns such as ease of use, data availability, speed, flexibity, etc. which are beyond the scope of this paper.

A useful metaphor can be drawn from the success of the "Value-at-Risk" framework in modeling market risk. Value-at-Risk has become the industry standard, with widespread use among end-users and acceptance as the basis for capital requirements among regulators. But in practice, Value-at-Risk encompasses a variety of different modeling techniques, as well as different parameter estimation techniques, some of which can be quite significant; for example, historical simulation vs. variance-covariance, delta-gamma vs exact Monte Carlo simulation, etc. It is the underlying coherence of the Value-at-Risk concept – that risk is measured by combining the relationship between the value of trading positions to market variables with the distribution of those underlying market variables – which ensures a consistency sufficient for widespread acceptance and regulatory change. In the same way, the underlying coherence of concept amongst these new sophisticated credit risk portfolio models should allow them to overcome differences in calculation procedures and parameter estimation. Rather than dissimilar competing alternatives, these models represent an emerging industry standard for credit risk management and regulation.

<u>References</u>

Carty, Lea and Dana Lieberman "Corporate Bond Defaults and Default Rates 1938 – 1995", Moody's Investors Service Global Credit Research, January 1996

Credit Suisse Financial Products "CreditRisk+ – A Credit Risk Management Framework", 1997

International Swaps and Derivatives Association "Credit Risk and Regulatory Capital", March 1998

Freund, John "Mathematical Statistics", 5th edition, Prenticel Hall, New Jersey, 1992

Gupton, Greg, Christopher Finger, and Mickey Bhatia "CreditMetrics Technical Document", Morgan Guaranty Trust Co., 1997

Kealhoffer, Stephen "Managing Default Risk in Derivative Portfolios" in *Derivative Credit Risk: Advances in Measurement and Management*, Renaissance Risk Publications, London, 1995

Kuritzkes, Andrew "Transforming Portfolio Management" *Banking Strategies*, July-August 1998, p. 57-62

Merton, Robert "On the Pricing of Corporate Debt: The Risk Structure of Interest Rates" *Journal of Finance*, vol. 29, 1974

Roberts, Dylan and James Wiener "Handle with Care", Oliver, Wyman & Company working paper, 1998

Vasicek, Oldrich "Probability of Loss on Loan Portfolio", KMV Corporation, 12 February 1987

Stuart, Alan and J. Keith Ord "Kendall's Advanced Theory of Statistics – Volume 1: Distribution Theory" 6th edition, John Wiley & Sons, New York, 1994

Wilson, Tom "Portfolio Credit Risk", *Risk*, September 1997 (part I) and October 1997 (part II)