Efficient Risk/Return Frontiers for Credit Risk

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The risk/return trade-off has been a central tenet of portfolio management since the seminal work of Markowitz [1952]. The basic premise, that higher (expected) returns can only be achieved at the expense of greater risk, leads naturally to the concept of an efficient frontier. The efficient frontier defines the maximum return that can be achieved for a given level of risk or, alternatively, the minimum risk that must be incurred to earn a given return. Traditionally, market risk has been measured by the variance (or standard deviation) of portfolio returns, and this measure is now widely used for credit risk management as well. For example, in the popular CreditMetrics methodology (J.P. Morgan [1997]), the standard deviation of credit losses is used to compute the marginal risk and risk contribution of an obligor. Kealhofer [1998] also uses standard deviation to measure the marginal risk and, further, discusses the application of mean-variance optimization to compute efficient portfolios. While this is reasonable when the distribution of gains and losses is normal, variance is an inappropriate measure of risk for the highly skewed, fat-tailed distributions characteristic of portfolios that incur credit risk. In this case, quantile-based measures that focus on the tail of the loss distribution more accurately capture the risk of the portfolio. In this article, we construct credit risk efficient frontiers for a portfolio of bonds issued in emerging markets, using not only the variance but also quantile-based risk measures such as expected shortfall, maximum (percentile) losses, and unexpected (percentile) losses.

In an earlier article (Mausser and Rosen [1999]), we considered several scenario optimization models for credit risk with a particular emphasis on the expected regret measure. A primary motivation for using this measure is the fact that expected regret is both relevant and tractable. Relevant measures capture key properties of the loss distribution while tractable measures can be optimized using computationally efficient methods such as linear programming. In contrast to expected regret, variance is generally not a relevant credit risk measure, while maximum losses and unexpected losses are relevant but not tractable. Arvanitis et al. [1998] present a useful example of these concepts: they show that the minimum-variance portfolio is far from efficient with respect to unexpected losses. However, given that the latter measure is not tractable, they can only approximate the efficient frontier by evaluating randomly generated portfolios.

Efficient frontiers for expected regret can be readily constructed, as demonstrated in Dembo [1998] and Dembo and Rosen [1999], for example. Furthermore, Mausser and Rosen [1999] showed that minimizing expected regret effectively reshapes the loss distribution, typically producing significant reductions in other risk measures as well. This suggests an attractive alternative to random search for constructing (approximate) efficient frontiers for intractable risk measures: use a tractable mea-
sure as a proxy, find efficient portfolios and then evaluate the resulting portfolios in terms of the desired risk measure. Furthermore, one can vary model parameters, or even the proxy risk measure itself, in an effort to obtain portfolios that are more efficient with respect to the intractable risk measure. In this manner, one can construct an empirical efficient frontier that acts as a useful approximation to the true efficient frontier.

Recently, Uryasev and Rockafellar [1999] developed a novel linear programming formulation to minimize expected shortfall, thereby showing it to be a tractable risk measure. In this article, we use both expected regret and expected shortfall to construct empirical efficient frontiers for maximum (percentile) losses and unexpected (percentile) losses. We also present a simple procedure, requiring the solution of a single linear program, that rebalances an arbitrary portfolio in order to reduce the maximum losses. Applying this procedure to portfolios that are optimal for expected regret and/or expected shortfall allows us to obtain improved empirical efficient frontiers for maximum losses.

This article is organized as follows. First, we briefly review several standard credit risk measures. We then describe the general approach for constructing an efficient frontier for an arbitrary risk measure. Next, we apply the methodology to a portfolio of emerging markets bonds and examine exact and empirical efficient frontiers for various risk measures. We also provide a simple procedure for reducing the maximum losses for an existing portfolio and use this to obtain improved empirical efficient frontiers. Finally, we offer our conclusions and recommendations for further study. The mathematical formulations used to obtain efficient frontiers for various risk measures appear in the appendix.

**Credit Risk Measures**

Exhibit 1 shows a sample portfolio credit loss distribution and several standard credit risk measures. Losses are calculated relative to the exposure that would exist, at a given time horizon, if all obligors maintained their current credit state. Since credit events (i.e., default or a change in credit rating) are relatively rare, the peak of the loss distribution is typically at or near zero. Note that the loss distribution is highly skewed; the long right tail reflects the infrequent, but substantial, losses that can occur when an obligor defaults. Conversely, negative losses (i.e., gains) can result from a net improvement in the credit ratings of obligors.

The expected losses equal the mean of the loss distribution. The maximum losses represent the largest loss that is anticipated to occur with a given probability. For example, the maximum losses at the 99th percentile level, denoted MaxLoss (99%), is likely to be exceeded only 1% of the time (the MaxLoss corresponds to the Value-at-Risk, or VaR, typically used to measure market risk). The unexpected losses, or CreditVaR, is the difference between the maximum and expected losses. Note that the expected and unexpected losses determine the amount of credit and capital reserves, respectively, required to support the portfolio. Expected shortfall, also known as tail conditional expectation or conditional VaR, measures the average size of a loss given that it exceeds the MaxLoss. Expected shortfall has been proposed as an alternative to Value-at-Risk since it exhibits more desirable theoretical properties than VaR (Artzner et al. [1998]). One attractive feature of expected shortfall is that it considers the entire tail of the loss distribution (beyond the MaxLoss), rather than just one point on the loss distribution. Thus, expected shortfall is more likely to draw attention to large losses in the extreme tail of the distribution than MaxLoss and CreditVaR, both of which effectively ignore losses beyond the specified quantile level.

Expected regret (Dembo [1991]) is measured with respect to a benchmark, whose value may be scenario-dependent. In this article, as in Mausser and Rosen [1999], we calculate expected regret as the expectation of losses that exceed some fixed threshold \( K \). Expected regret differs from expected shortfall in that the threshold \( K \) is prespecified and need not equal the MaxLoss, and in its use of the unconditional, rather than the conditional, expectation.
CONSTRUCTING THE EFFICIENT FRONTIER

We construct efficient frontiers for credit risk by solving optimization problems of the form:

\[
\begin{align*}
\text{minimize:} & \quad \text{risk} \\
\text{subject to:} & \quad \text{expected return} \geq R \\
& \quad \text{trading constraints}
\end{align*}
\]

(1)

The variables in the problem above are the sizes, or weights, of the positions in the portfolio. Since it is generally not possible to enter into positions of arbitrary size, perhaps due to market liquidity and/or budget limitations, the trading constraints ensure that the optimal position sizes are in fact reasonable. Thus, Equation (1) rebalances a portfolio, subject to specified trading constraints, in order to obtain an expected return of at least \( R \) while incurring the smallest amount of risk. The appendix contains the mathematical formulations of Equation (1) for the following risk measures: variance, expected regret, expected shortfall, MaxLoss and CreditVaR. Note that as an alternative to Equation (1), it is also possible to maximize the expected return subject to not exceeding a specified level of risk. For optimization purposes, it is generally preferable to use a formulation in which the non-linearities, if any, are contained in the objective function (e.g., when risk is measured in terms of variance, which is a non linear function of position size, Equation (1) is the preferred formulation).

Since the return is effectively traded off with risk, it is important to measure these two quantities in a consistent manner when constructing efficient frontiers. Consider, for example, a security \( i \) that is subject to credit risk. The market-driven return (\( r_i \)) of security \( i \), assuming no change in credit state, typically exceeds the risk-free rate (\( r_f \)) (in this article, \( r_f \) is obtained by valuing securities at a given time horizon using the corresponding forward curve). The excess return, or "spread," represents a risk premium that compensates investors for potential losses due to credit events (and possibly illiquidity). From Exhibit 1 it follows that a credit loss has both an expected and an unexpected component: the former is simply the mean of the loss distribution, or the expected losses, while the latter is the amount by which the loss differs from the mean. While MaxLoss and expected shortfall consider the entire loss (i.e., they are measured with respect to the benchmark, or zero), the variance and CreditVaR deal only with the unexpected portion of the loss (i.e., they are measured relative to the mean). Therefore, to represent the risk/return trade-off in Equation 1 properly, we decompose the spread into expected and unexpected components, denoted \( r_i' \) and \( r_i'' \) respectively. When constructing efficient frontiers, we trade off the market-driven return of security \( i \):

\[
r_i = r_f + r_i' + r_i''
\]

(2)

with expected regret, MaxLoss and expected shortfall, and use the expected return:

\[
r_i' = r_f + r_i''
\]

(3)

which is net of expected credit losses, to offset variance and CreditVaR. Note that only the components of the spread (i.e., \( r_i' \) and \( r_i'' \)) actually compensate for the respective risks in Equations (2) and (3). We include the risk-free rate when constructing efficient frontiers simply for consistency with observed returns; since \( r_f \) is the same for all securities, this has no effect on the composition of efficient portfolios.

For example, consider a credit instrument \( i \) with a mark-to-market value of U.S.$100 that will be worth U.S.$108 at some future time if the issuer maintains its current credit rating. The market-driven return for this instrument is \( r_i = 8\% \). Suppose that the expected future value of the instrument is U.S.$107 if potential credit changes are taken into account. In this case, the expected loss due to credit events is U.S.$1 and the corresponding return that compensates for this expected loss is \( r_i'' = 1\% \).

\begin{table}[h]
\centering
\begin{tabular}{|l|c|}
\hline
\textbf{Statistics for One-Year Loss Distribution (Millions of USD)} & \\
\hline
\textbf{Expected losses} & 95 \\
\textbf{Standard deviation} & 232 \\
\textbf{Maximum losses (99\%)} & 1,026 \\
\textbf{CreditVaR (99\%)} & 931 \\
\textbf{Expected shortfall (99\%)} & 1,320 \\
\textbf{Maximum losses (99.9\%)} & 1,782 \\
\textbf{CreditVaR (99.9\%)} & 1,687 \\
\textbf{Expected shortfall (99.9\%)} & 1,998 \\
\hline
\end{tabular}
\end{table}
If the risk-free rate is \( r_f = 5\% \), then it follows from Equation (2) that \( \bar{r}_u = 2\% \) compensates for the unexpected loss. For this instrument, therefore, the expected return that is traded off with variance and CreditVaR is \( \bar{r}' = 7\% \).

As the required return (\( R \)) increases, so does the minimum amount of risk; together, these optimal risk/return trade-offs define the efficient frontier. Parametric programming can be used to solve Equation (1) in response to continuous changes in \( R \). However, for simplicity, in this article we solve Equation (1) for a finite set of returns and then construct the efficient frontier by interpolating linearly between the resulting points. The potential interpolation errors can be made arbitrarily small by increasing the number of points.

If a risk measure is intractable, solving Equation (1) may not be possible in practice. For example, minimizing MaxLoss gives rise to an integer programming problem that contains as many integer variables as there are scenarios. Given the thousands of scenarios required in a typical credit simulation, such a problem may take days, if not longer, to solve even on the fastest computers. In this case, it is necessary to construct an empirical efficient frontier that approximates the true efficient frontier. As noted previously, this can involve a brute-force approach that randomly generates and evaluates thousands of portfolios (Arvinitis et al. [1998]) or a more structured approach that uses a tractable risk measure as a proxy in Equation (1) in order to obtain portfolios that are “close to” optimal for the desired risk measure. The latter option, which will be used in this article, represents one example of what is generally known as a heuristic strategy. Other methods, such as problem-specific heuristics that attempt to find good solutions to Equation (1) for intractable risk measures directly, rather than by means of a proxy measure, are also possible.

CASE STUDY

To illustrate the methodology, we consider a portfolio of long-dated corporate and sovereign bonds issued in emerging markets, originally described in Bucay and Rosen [1999]. The portfolio, with a mark-to-market value of U.S.$8.8 billion as of October 13, 1998, consists of 197 emerging markets bonds, issued by 86 obligors in 29 countries. Instruments are denominated in U.S.$, except for 11 fixed-rate bonds, which are denominated in seven other currencies. Bond maturities range from a few months to 98 years with an average maturity of 9.6 years.

Credit events include both default and credit migration over a one-year time horizon. Credit migration probabilities are obtained from S&P transition matrix as of July 1998 (S&P's 1998). Recovery rates, in the event of default, are assumed to be constant and equal to 30% of the risk-free value for all obligors except two, which have lower rates. Asset correlations are driven by a multifactor model through a set of country and industry indices chosen from the CreditMetrics dataset, and through a specific volatility component. Obligors are mapped to indices as described in Bucay and Rosen [1999]. The portfolio's loss distribution is created from a Monte Carlo simulation of 20,000 scenarios on joint credit states (market risk factors are not included in the simulation). Exhibit 2 summarizes several risk measures derived from the one-year credit loss distribution of the portfolio.

We assume that the current holdings correspond to a weight of one for each obligor. Thus, the decision variables for optimization purposes essentially represent multiples of the current holdings (e.g., a weight of two implies that the current position is doubled, for instance). More generally, the decision variables may also refer to the weights of sector classes or to positions in individual instruments, depending on the level of the analysis (Mausser and Rosen [1999]). The trading constraints, enforced when constructing efficient frontiers, include:

- the current mark-to-market value of the portfolio must be maintained,
- no short positions are allowed,
- the long position in the debt of an individual obligor cannot exceed 20% of the (current) portfolio value.

We now examine efficient frontiers for the emerging markets bond portfolio under a variety of risk measures. In doing so, we also evaluate the relative performance of tractable risk measures such as variance, expected regret, and expected shortfall, in constructing empirical efficient frontiers. Before that, it is useful to examine an empirical efficient frontier in detail.

Example of an Empirical Efficient Frontier

Exhibit 3 shows the efficient frontier for expected regret with a threshold of U.S.$100 million. If we plot the MaxLoss (99%) of the portfolios that are efficient with respect to expected regret, then we obtain the curve shown in Exhibit 4. Observe, for example, that the effi-
cient portfolio returning 7.5% in Exhibit 3 incurs an expected regret of U.S.$10 million and this same portfolio has a MaxLoss (99%) of U.S.$283 million.

We can obtain an empirical frontier for MaxLoss (99%) by solving the expected regret model for various thresholds and then, for a given return, selecting the smallest MaxLoss (99%) among all of the efficient portfolios. This concept is illustrated in Exhibit 5, which plots the MaxLoss (99%) of efficient portfolios that derive from solving the expected regret model for thresholds of (in millions U.S.$) $K = 100, 200, 300, 400, 500, 600,$ and $700$. The empirical efficient frontier obtained from the expected regret model, shown in bold, is defined by the outer envelope of these curves. Note how different thresholds contribute various sections of the frontier, ranging from $K = 100$ at low return levels to $K = 600$ at the highest levels (in general, a threshold of $K = K_0$ contributes the section of the empirical frontier that corresponds to a MaxLoss (99%) of $K_0$). A threshold of $K = 700$ is too high to contribute a portion of the empirical frontier for returns of 10% or less in this case. These observations are consistent with the results reported in Luo and Mausser [1999] and Mausser and Rosen [1999], which suggest that higher thresholds are generally more effective for reducing the extreme tail of the loss distribution (obtaining higher returns typically results in a longer-tailed distribution).

In the following examples, all empirical efficient frontiers based on the expected regret model are constructed in the manner shown in Exhibit 5 (i.e., solve the model using the seven different thresholds, evaluate the efficient portfolios in terms of the desired risk measure, and select the best of these values for each level of return).

The empirical efficient frontier based on expected shortfall is constructed in a similar fashion, except that the model parameter is the quantile level, rather than the loss threshold, in this case. Specifically, the expected shortfall model is solved for quantile levels 95%, 96%, 97%, 98%, 99%, and 99.9%. In contrast, the variance model is solved only once for each level of return since it does not contain any parameters.

**Variance Efficient Frontier**

Let us first examine the efficient frontier for the variance (to facilitate comparisons with other measures, we report the standard deviation rather than the variance.

![Exhibit 5](image.png)

**E X H I B I T 5**
Example Empirical Efficient Frontier

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**E X H I B I T 3**
Expected Regret ($K = 100$) Efficient Frontier

**E X H I B I T 4**
MaxLoss (99%) of Efficient Portfolios for Expected Regret ($K = 100$)
itself. Although variance is not a relevant measure of credit risk, it is nevertheless a common measure that will be interesting for comparison purposes. Exhibit 6 shows the exact efficient frontier, obtained by solving the variance minimization model in the appendix, as well as the empirical frontiers derived from expected regret and expected shortfall. Note that while expected regret and expected shortfall are compensated by the market-driven returns (and their efficient frontiers are constructed accordingly), the corresponding empirical frontiers in Exhibit 6 report the expected returns of the portfolios.

The original portfolio, with a standard deviation of U.S.$232 million and an expected return of 6.12%, is clearly inefficient and can be improved using any of the three models. For example, the minimum-variance portfolio returns approximately 8.15% for the same level of risk as the original portfolio. As expected, the exact efficient frontier dominates the empirical efficient frontiers. Although Exhibit 6 may suggest that expected regret is a better proxy than expected shortfall in this case, it is necessary to interpret such results with caution, since they reflect only the particular values of the model parameters used in this study.

As mentioned previously, the highly skewed, fat-tailed distributions characteristic of credit losses imply that variance inadequately measures a portfolio’s credit risk. This is evident if one compares the expected regret, expected shortfall, and variance models when constructing empirical efficient frontiers for quantile-based risk measures.

### Expected Shortfall Efficient Frontier

Exhibit 7 shows the exact efficient frontier for expected shortfall (99%) along with the empirical frontiers obtained from the expected regret and variance models. All three models improve upon the original portfolio, which earns a market-driven return of 7.26% and has an expected shortfall (99%) of U.S.$1,320 million. By comparison, an efficient portfolio returning 7.26% incurs an expected shortfall (99%) of only U.S.$264 million. While the empirical frontier based on expected regret provides a good approximation to the exact efficient frontier, the variance model performs rather poorly in this case. For example, at the 7.26% level of return, the portfolio derived from the expected regret model has an expected shortfall (99%) of U.S.$265 million (i.e., 0.4% larger than the minimum). In contrast, the variance-minimizing portfolio has an expected shortfall (99%) of U.S.$846 million, which is almost 84% larger than the minimum.

### MaxLoss Efficient Frontier

We now consider an intractable measure, the (percentile) MaxLoss, for which the exact efficient frontier cannot be obtained easily (as described in the appendix, optimizing MaxLoss requires solving an integer program). Exhibit 8 shows three empirical efficient frontiers for MaxLoss (99%); while a comparison with the exact efficient frontier is not possible, the empirical frontiers, particularly those obtained from the expected regret and
expected shortfall models, show that the original portfolio can be substantially improved. For example, the MaxLoss (99%) can be reduced from U.S.$1,026 million to U.S.$184 million while maintaining a market-driven return of 7.26%. Note that the minimum-variance portfolios perform significantly worse than those of the expected regret and expected shortfall models, consistent with the results of Arvinitis et al. [1998]. The variance-minimizing portfolio that returns 7.26%, for example, incurs a MaxLoss (99%) of U.S.$367 million, which is almost double that of the portfolios derived from the other two models.

Given an arbitrary portfolio (in particular, one that is on the empirical efficient frontier for expected regret or expected shortfall), it is possible to rebalance the positions in order to reduce its MaxLoss. The following heuristic procedure, which requires solving a linear program, allows us to obtain an improved empirical efficient frontier that dominates those of the aforementioned risk measures.

Recall that the MaxLoss $100(1 - \alpha)\%$ is the size of the loss that will be exceeded only $100\alpha\%$ of the time [e.g., for the MaxLoss (99%), $\alpha = 0.01$]. Consider an arbitrary portfolio that has been simulated over a set of $m$ scenarios. Based on the simulation, the portfolio’s MaxLoss $100(1 - \alpha)\%$ equals the loss $L_k$ in some scenario $k$, where the total probability of all scenarios with losses greater than or equal to $L_k$ is at least $\alpha$. Keeping track of this “tail” probability in an optimization problem requires integer variables [see Equation (A-4) in the appendix]. However, if we fix the scenarios that are allowed to exceed the MaxLoss in advance, and ensure that their total probability is less than $\alpha$, then minimizing the largest loss among the remaining scenarios provides an estimate (specifically, an upper bound) for the true minimum MaxLoss $100(1 - \alpha)\%$. This latter problem only requires solving a linear, rather than an integer, program.

Exhibit 9 illustrates the basic idea underlying the heuristic procedure. Consider a portfolio that has been simulated over a set of 100 equally weighted scenarios. The MaxLoss (95%) equals the fifth-largest loss, which corresponds to the loss of 1,000 in scenario 11 in this case. We fix the four “tail” scenarios that are allowed to exceed the MaxLoss (95%) (i.e., scenarios 73, 82, 16, and 21), and then minimize the largest loss among the remaining scenarios by solving a linear program. As shown in Exhibit 9, this reduces the MaxLoss (95%) to 981, which occurs in scenario 64.

Naturally, the quality of the improved solution depends on picking the right set of tail scenarios a priori, which is by definition, an intractable problem. Thus, in the example shown in Exhibit 9, it may well be possible to further reduce the MaxLoss (95%) if four different tail scenarios are selected. However, even a small set of 100 scenarios gives rise to almost four million distinct combinations of tail scenarios in this case! The heuristic procedure is built on the assumption that minimizing the MaxLoss for a “good” preselected set of tail scenarios will result in a MaxLoss that is close to optimal. Since a portfolio that is on the empirical efficient frontier has a relatively small MaxLoss to begin with, fixing the tail scenarios to be those that exceed its MaxLoss is a reasonable choice.

Exhibit 10 shows the empirical efficient frontiers for MaxLoss (99%) that derive from expected regret and expected shortfall, along with a heuristically improved empirical frontier. The curve denoted “MaxLoss Heuristic” is obtained by applying the above procedure to the best available solution [i.e., the one having the smallest MaxLoss (99%) for a given level of return] among the expected regret and expected shortfall models. The heuristic further reduces the MaxLoss (99%) by an average of 1.9% and a maximum of 4.7%. For a return of 7.26%, for example, the MaxLoss (99%) is reduced from U.S.$184 million to U.S.$179 million.

The heuristic procedure has a more significant impact on the empirical efficient frontier for MaxLoss (99.9%), as shown in Exhibit 11. In this case, the average reduction in MaxLoss (99.9%) is 6.9% and the largest improvement is 21%.
To ascertain the difficulty of solving the MaxLoss minimization problem directly, we performed several numerical experiments using the branch and bound algorithm implemented in the CPLEX Mixed Integer Solver (ILOG 1998). Starting from an existing empirically efficient portfolio (i.e., one produced by the above heuristic procedure), we were unable to obtain a solution with a lower MaxLoss after 12 hours of computation time in any of the cases considered. In contrast, linear programs for the expected regret and expected shortfall models are typically solved in several minutes on the same machine.

**CreditVaR Efficient Frontier**

The CreditVaR, which equals the unexpected losses, is another example of an intractable risk measure. Consistent with our previous results for quantile-based risk measures, the empirical efficient frontiers for CreditVaR (99.9%) that derive from the expected regret and expected shortfall models clearly dominate the empirical frontier obtained from minimizing variance (Exhibit 12). For example, the variance-minimizing portfolio that returns 7.5% has a CreditVaR (U.S.$764 million), which is more than double the CreditVaR of portfolios obtained from the expected regret...