Credit Risk Modeling: A General Framework

Ren-Raw Chen*
Rutgers University

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ABSTRACT

The two well-known approaches for credit risk modeling, structural and reduced form approaches, have their advantages and disadvantages. Due to the fundamentally different assumptions of the two approaches, the structural models are used for default prediction that focuses on equity prices and reduced form models are used for credit derivatives pricing that focuses on debt values. In this paper, via a simple discrete binomial structure, we provide a unified view of the two approaches. In particular, in our formulation, the pricing formulas for risky debts are identical under the two approaches. The two approaches differ in only the recovery assumption. This result makes comparison of various models empirically possible. We demonstrate, in a credit derivative example that is sensitive to the recovery assumption, how different recovery assumptions impact its prices.

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1 INTRODUCTION

There have been two well-known approaches, structural and reduced form, for credit risk modeling. Reduced form models, represented by Jarrow and Turnbull (1995) and Duffie and Singleton (1997, 1999) assume defaults (or credit events) occur exogenously (usually by a Poisson process) and a separately specified recovery is paid upon default.[1] Structural models, on the other hand, assume defaults occur when the value of the firm falls below a certain default point and a certain recovery is paid. The "true structural" model of Geske (1977) and Geske and Johnson (1984) assumes the default point to be the market value of debt that is endogenously computed and the firm value is the recovery. This formulation leads to multi-variate probability functions. The "barrier structural" models, pioneered by Black and Cox (1976), obtain uni-variate valuation formulas by assuming an exogenous default point and an exogenous recovery amount.[2]

Structural models for credit risk modeling have been mainly used for default prediction[3] or capital structure analysis while reduced form models are mainly used by investment banks to price credit derivatives.[4] This is because structural models rely on the information from equity prices while reduced form models from debt prices. Furthermore, reduced form models are more computationally efficient due to their exogenous default and recovery assumptions, which are important for pricing credit derivatives. Another reason for the reduced form models to be chosen by investment banks is its relative ease to incorporate the term structure of the default free interest rates.[5]

In this paper, we clarify the difference between the "true structural" model of Geske (1977) and Gseke and Johnson (1984) in which defaults occur when the (market) value of the firm falls below the (market) value of debt and the "barrier structural model" in which defaults occur when the value of the firm crosses an exogenously pre-defined barrier. The former valuation leads to multi-variate distributions while the latter is a univariate valuation.[6] We then show that reduced-form and structural models can be made consistent under a simple discrete binomial formulation. Under this binomial formulation, the two sets of models have identical pricing formulas for risky debts. The only difference is different recovery

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[1] Other reduced form models include, among others, Duffie and Lando (1997), Jarrow, Lando, and Turnbull (1997), Jarrow and Yu (2001), Lando (1998), Madan and Unal (2000), and Schönbucher (1998).
[2] Other structural models include, for example, Anderson and Sundaresan (1996), Leland and Toft (1996), Longstaff and Schwartz (1995), Zhou (2001), and Bélanger, Shreve, and Wong (2002). Readers can also find a more thorough survey by Uhrig-Homburg (2002).
[3] For example, see KMV's EDF and Moody's RiskCalc.
[4] It should be noted that structural models are also proposed for pricing risky bonds. However, they have never gained support from the industry due to the difficulty in calibration.
[5] The two most known reduced form models by Jarrow and Turnbull (1995) and Duffie and Singleton (1999) can both be easily incorporated into existing term structure models.
[6] Bélanger, Shreve, and Wong (2002) provide a unified model that nests all barrier structural models. This paper can be viewed as an extension of their work.

assumptions. This result makes it possible to compare various models. In an application, we use credit default swaps to examine the impact of different recovery assumptions.

The remaining of the paper is organized as follows. Section 2 lays out the basic valuation equations. In particular, we specify the forward measure technique to incorporate stochastic interest rates. Section 3 presents the binomial framework under which all models share the same valuation formula. In Section 4, we discuss calibration issues of various models and perform model comparison. Section 5 uses the models discussed to value the most popular credit derivative contract – default swaps. Finally, the paper is concluded in Section 6.

2. BASIC SETUP[7]

Define a T-maturity default-free pure discount bond price at current time t, denoted $P(t,T)$ with $P(T,T)=1$ for all T. Also define a set of dates: $0=T_0<T_1<\cdots<T_{N-1}<T_N$. For simplicity and without loss of generality, we also assume $h=T_i-T_{i-1}$ for all $i=1,\ldots,N$. Let $\ell(T_i,T_j)$, $0\le i<j\le N$, represent the default free "term" interest rate (annualized) over $[T_i,T_j]$. Hence, by definition:

$$(1)\qquad \ell(T_i,T_j)=\frac{1}{T_j-T_i}\left(\frac{1}{P(T_i,T_j)}-1\right)$$

Denote by $\mathbb{Q}$ the risk neutral measure under which defined the instantaneous short rate r. Then we have:

$$(2)\qquad P(t,T)=E_t^{\mathbb{Q}}[\Lambda(t,T)]$$

where

$\Lambda(t,s)=\exp\left(-\int_t^s r(u)du\right)$, $0\le t<s\le T$, and

$E_t^{\mathbb{Q}}[\cdot]$ represents the expectation conditional on the information set at time t under the risk neutral measure $\mathbb{Q}$.

For the sake of convenience, we further define T_n-forward measure, denoted by $\mathbb{F}_n$, to be equivalent to the measure $\mathbb{Q}$ and the corresponding Radon-Nikodym derivative is given by:[8]

$$(3)\qquad \frac{d\mathbb{F}_n}{d\mathbb{Q}}=\frac{\Lambda(t,T_n)}{P(t,T_n)}$$

[7] This section and part of the next section originally derived in Chen and Huang (2001).
[8] For the forward measure, see, for example, Jamshidian (1987) and Hull (2000).

Under (3) the forward price is merely the forward expectation of the bond price:

(4) $$F(0,t,s) = E_0^{\mathsf{F}_t}[P(t,s)] = \frac{P(0,s)}{P(0,t)}$$

and so is the discrete forward rate:

(5) $$E_0^{\mathsf{F}_t}[\ell(u,t)h] = \frac{1}{F(0,u,t)} - 1$$

Finally, we reiterate the separation of expectation using the forward measure:

(6) $$\begin{aligned} E_0^{\mathsf{Q}}[\Lambda(0,t)I_{\{t<\tau\}}] &= E_0^{\mathsf{Q}}[\Lambda(0,t)]E_0^{\mathsf{F}_t}[I_{\{t<\tau\}}] \\ &= P(0,t)Q(0,t) \end{aligned}$$

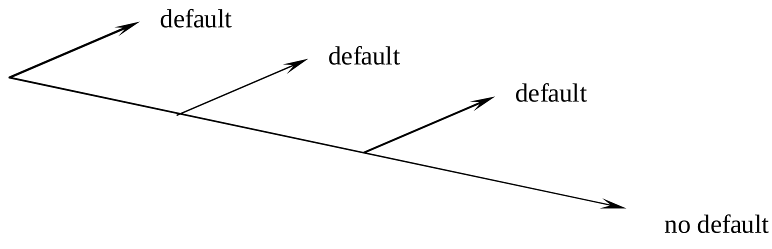
where τ is the default time and:

(7) $$Q(0,t) = E_0^{\mathsf{F}_t}[I_{\{t<\tau\}}]$$

as the survival probability under the forward measure where $I_{\{\cdot\}}$ is an indicator function and τ is the default time. Equation (6) represents the present of $1 paid if there is no default and 0 otherwise. Hence, it is also known as the risky discount factor.

3 THE UNIFIED FRAMEWORK

The unified model is a simple binomial model that defines default and no default states at every period. This model is general to accommodate any form of recovery upon default and any form of cash flows under no default.



Hence, their solution to the risky fixed rate coupon bond can be written as:

$$V(0,T_n) = E_0^{Q}\left[\sum_{i=1}^{n} \Lambda(0,T_i)c_n h I_{\{\tau>T_i\}} + \Lambda(0,T_n)I_{\{\tau>T_n\}} + \Lambda(0,\tau)w_n(A(\tau),r(\tau))I_{\{\tau<T_n\}}\right] \tag{8}$$
$$= \sum_{i=1}^{n} P(0,T_i)Q(0,T_i)c_n h + P(0,T_n)Q(0,T_n) + R_n(0)$$

where c_n is the fixed coupon, $w_n(\cdot,\cdot)$, which is a function of some state variable *A* and interest rate *r*, represents the recovery amount upon default, and $R_n(0)$ is the current value of expected recovery. Note that the first two terms of the second line follows directly form (6) and (7). They represent the value of coupons and face value. The last term represents the current recover value, which is an expected present value of the recovery amount upon default, a function of some state variable(s), *A* and the risk free interest rate, *r*.

Chen and Huang (2001) were the first to make the observation that the binomial process is the general framework to incorporate all credit risk models. As a result, they can use (8) to derive general upper and lower bounds for credit spreads.[9] However, they still take recovery as exogenously given. In this paper, we focus on the models that have endogenous recovery. In particular, we study the Geske model (1977) that has an endogenous recovery process. We compare the Geske model with other structural models and reduced form models by using the binomial default process given above. The binomial default process allows us to compare various models via only the recovery assumptions made by various models.

The binomial default process assumes that defaults can occur only at discrete points (i.e. coupon payment times). This is not unreasonable because usually companies do not have to declare default unless they fail a payment of interest or the solvency test by regulators or creditors, both of which happen in discrete time. So far, we have not used any model specification. The remainder of this section presents $V(0,T_n)$ in various model specifications.

A. The Jarrow-Turnbull Model

The discrete binomial model is most straightforward to explain the Jarrow-Turnbull model because both default event and recovery are exogenously specified in the mdoel. The Jarrow-Turnbull model assumes that a fixed recovery is paid at maturity regardless of the time of default. Hence, the closed form solution exists for the coupon bond as follows:

$$V_{\mathrm{JT}}(0,T_n) = \sum_{i=1}^{n} P(0,T_i)Q(0,T_i)c_n h + P(0,T_n)Q(0,T_n) + w_n P(0,T_n)[1-Q(0,T_n)] \tag{9}$$

[9] Chen and Huang (2001) also present the pricing formula for the floating rate bond. Since the difference between fixed and floating rate bonds is only due to the coupon, we focus on the fixed rate bond in this paper.

In such a case, $R_n(0) = w_n P(0,T_n)[1 - Q(0,T_n)]$. This model is also recognized as "recovery of proportional to par". In the Jarrow-Turnbull model, default events are assumed to follow a Poisson process that gives the survival probability:[10]

(10) $$Q(0,t) = E_0^{\mathrm{F}_t}\left[\exp\left(-\int_0^t \lambda(u)du\right)\right]$$

where $\lambda(t)$ represents the hazard rate (intensity) of the Poisson process. Default occurs when there is a jump with a probability $\lambda(t)dt$ over the period *dt*.

An extended Jarrow-Turnbull model of the following is usually used in the industry in which defaults are allowed only at coupon times and a fixed recovery is paid upon default.

(11) $$V_{\mathrm{JT}}(0,T_n) = \sum_{i=1}^{n} P(0,T_i)Q(0,T_i)c_n h + P(0,T_n)Q(0,T_n) + w_n(T_i)P(0,T_i)[Q(0,T_{i-1}) - Q(0,T_i)]$$

In such a case, $R_n(0) = \Sigma_{i=1}^n w_n(T_i)P(0,T_i)[Q(0,T_{i-1}) - Q(0,T_i)]$. In many cases, continuous default is necessary, hence a continuous version of recovery is also commonly used: $R_n(0) = \int_0^{T_n} w_n(t)P(0,t)[-dQ(0,t)]$.

B. The Duffie-Singleton Model

Like the Jarrow-Turnbull model, the Duffie-Singleton model also assumes a Poisson process for defaults. Unlike the Jarrow-Turnbull model, the Duffie-Singleton model assumes that recovery is paid immediately upon default and equals a fraction of what the bond is worth immediately prior to default. In our formulation, it means:

(12) $$w_n(t) = \delta V(t,T_n)$$

where δ is the constant recovery ratio on the value of the bond prior to default. Substituting this result back to (8), we get:

(13) $$V_{\mathrm{DS}}(0,T_n) = \sum_{i=1}^{n} P(0,T_i)Q^*(0,T_i)c_n h + P(0,T_n)Q^*(0,T_n)$$

[10] If the hazard rate, λ, is a deterministic function, then the forward-measured-adjusted expectation reduces to $\exp(-\int_0^t \lambda(u)du)$. If λ is stochastic, then we should follow Lando (1998) and the standard forward measure technique to find the solution to the expectation. Under this situation, it is not clear that there will be a closed form solution to this expectation if the interest rate process and the hazard rate process are not Gaussian.

where c_n is the fixed coupon and

(14) $$Q^*(0,T_i) = E_0^{F_i}\left[e^{-(1-\delta)\int_0^{T_i} \lambda(t)dt} \right] = Q(0,T_i)^{1-\delta}$$

It can be seen that the Duffie-Singleton model cannot differentiate the survival probability from the recovery, a drawback of the model. However, the solution represented by (13) and (14) is a closed form solution of the affine style, more easily to derive closed form solutions when the recovery ratio and the intensity parameter are random.

It is clear that in the Duffie-Singleton model, recovery is blended into survival probabilities. In other words, recovery in the Duffie-Singleton model contains survival probabilities. Formally, we can write recovery as:

(15) $$R_n(0) = \sum_{i=1}^{n} P(0,T_i)Q(0,T_i)[M(0,T_i)-1]c_n h + P(0,T_n)Q(0,T_n)[M(0,T_n)-1]$$

where

$$M(0,t) = \tilde{E}_0^{\tilde{F}_t}\left[e^{\delta \int_0^t \lambda(u)du} \right] \text{ and}$$
$$\frac{dF_i}{d\tilde{F}_i} = \frac{e^{-\int_0^{T_i} \lambda(t)dt}}{Q(0,T_i)}$$

C. The Extended Merton (Barrier Structural) Model

The structural models can be traced back to Black and Scholes (1973) and Merton (1974) who observe that the company's equity is a European call option and hence the single-maturity-date debt contains default risk identical to a covered call. The recovery in the Black-Scholes-Merton model is therefore the firm value if default occurs at maturity. To extend the single period model of Black-Scholes and Merton, as mentioned earlier, there are two approaches. The "barrier structural" models assume an exogenous default barrier. Default is defined as the asset value crossing such a barrier.[11] The extension along this is pioneered by Black and Cox (1976), followed by Leland and Toft (1996) and Longstaff and Schwartz (1995), and recently extended by Bélanger, Shreve, and Wong (2002). Another is a "pure structural" model by Geske (1977) and Geske and Johnson (1984) who adopt the compound option approach and treat default as the inability of the company to fulfill its debt obligations (i.e. negative equity value).

[11] The barrier-based structural models are particularly popular in industry. See, for example, KMV (recently acquired by Moody's) and CreditGrades.

In this sub-section, we include multi-period debt structure in the Merton model. We assume defaults occur if the firm fails to meet its coupon obligations at any give time. Since the coupon obligations are exogenously given, the "barrier structural" models can be viewed as the continuous time limit of what is described here.

Consistent with our discrete setup, let $X_1, \quad , X_n$ be the series of external barriers, crossing which by the firm value represents default. These discrete time barriers can be interpreted as cash obligations at each time and failure to make the obligation results in default of the firm. These cash obligations can be regarded as a series of zero coupon bonds issued by the firm. The values of assets and debts (zero coupon) are labeled as $A(t)$, and $D(t, T_n)$ respectively for an arbitrary *t*. Default is defined as $A(T_i) < X_i$ at time T_i. To obtain closed form solutions, we assume the continuity of $A(t)$, as by Geske (1977) and Geske and Johnson (1984). To arrive at closed form results, we need to assume a log normal process for the firm's asset value and a normal process for the instantaneous short rate:

$$\begin{bmatrix} \dfrac{dA(t)}{A(t)} \\ dr(t) \end{bmatrix} = \begin{bmatrix} r(t) \\ \mu(r,t) \end{bmatrix} dt + \begin{bmatrix} \sigma_A & 0 \\ 0 & \sigma_r \end{bmatrix} \times \begin{bmatrix} \sqrt{1-\rho_{Ar}^2} & \rho_{Ar} \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} dW_A^{\mathrm{Q}}(t) \\ dW_r^{\mathrm{Q}}(t) \end{bmatrix} \tag{16}$$

where σ_A, σ_r, and ρ_{Ar} are constants, and $dW_A^{\mathrm{Q}} dW_r^{\mathrm{Q}} = 0$. From (16), we know that the correlation between the (log) asset value and the interest rate is ρ_{Ar}. Also note that both $W_A^{\mathrm{Q}}(t)$ and $W_r^{\mathrm{Q}}(t)$ are independent Wiener processes under the Q measure. Applying Ito's lemma to (2) in Section 2, we obtain that:

$$\begin{aligned} \frac{dP(t,T)}{P(t,T)} &= \frac{1}{P(t,T)} \left[\frac{\partial P(t,T)}{\partial r} \mu(r,t) + \frac{1}{2} \frac{\partial^2 P(t,T)}{\partial r^2} \sigma_r^2 + \frac{\partial P(t,T)}{\partial t} \right] dt + \frac{1}{P(t,T)} \frac{\partial P(t,T)}{\partial r} \sigma_r dW_r^{\mathrm{Q}}(t) \\ &\equiv r(t)dt + \sigma_P(r,t,T) dW_r^{\mathrm{Q}}(t) \end{aligned} \tag{17}$$

The Gaussian models that satisfy (17) are Vasicek (1977), Ho and Lee (1986), and Hull and White (1990), in all of which the diffusion term, $\sigma_P(r,t,T)$, is independent of the interest rate and can be written as $\sigma_P(t,T)$.

Then, we can derive the value of T_n-maturity debt is:

$$\begin{aligned} D(0,T_n) &= E_0^{\mathrm{Q}}[\Lambda(0,T_n) \min\{X_n, A(T_n)\}] \\ &= P(0,T_n) E_0^{\mathrm{F}_n}[\min\{X_n, A(T_n)\}] \qquad\qquad n \ge 1 \\ &= P(0,T_n) X_n \mathrm{N}_n^- + A(0)(\mathrm{N}_{n-1}^+ - \mathrm{N}_n^+) \end{aligned} \tag{18}$$

where

$$\mathrm{N}_j^{\pm} = N_j(h_1^{\pm}(X_1), \quad , h_j^{\pm}(X_j); \rho_{ik})$$
$$\rho_{ik} = \sqrt{T_i / T_k}, \quad \forall\, i < k < j$$
$$h_i^{\pm}(X_i) = \frac{\ln \frac{A(0)}{P(0,T_i)X_i} \pm \frac{1}{2} v^2(0,T_i)}{v(0,T_i)}$$
$$v^2(0,T_i) = \int_0^{T_i} \sigma_A^2 + \sigma_P^2(u,T_i) - 2\rho_{Ar}\sigma_A\sigma_P(u,T_i)du$$

where ρ_{ik} represents the auto-correlation between $\ln A(T_i)$ and $\ln A(T_k)$ which is $\sqrt{T_i / T_k}$ for $T_i < T_k$. The derivation of (18) is given in an appendix. In (18), $N_j(\quad)$ is a j-dimensional cumulative normal probability.[12] The derivation of (18) is given in an appendix that both N_n^+ and N_n^- are probabilities being in the money, only defined in different probability measures.

Equation (18) has an interesting interpretation. Note that N_n^- is the survival probability from now till T_n ($= Q(0,T_n)$) and $\mathrm{N}_{n-1}^- - \mathrm{N}_n^-$ (or $\mathrm{N}_{n-1}^+ - \mathrm{N}_n^+$ under a different measure) is the *unconditional* default probability between T_{n-1} and T_n. In other words, (18) implies that if default does not happen (with probability N_n^-), the bond receives X_n. If default does happen, it receives $A(T_n)$. Note that any prior default (default before T_{n-1}) should pay no recovery for such a bond since it is the most junior bond and the last to receive recovery. And this recovery is multiplied by the default probability and discounted to be $A(0)(\mathrm{N}_{n-1}^+ - \mathrm{N}_n^+)$ today ($= R_n(0)$).

It should be noted that (18) is not a closed form result unless $v(0,T_i)$ has a closed form expression. Rabinovitch (1989) shows that a closed form expression for $v(0,T_i)$ exists if the term structure model follows Vasicek (1977) and does not exist if it follows Cox-Ingersoll-Ross (1985). We restate the closed form result of Rabinovitch in an appendix.

The T_n-maturity coupon bond that pays hc_n as coupons can be regarded as the total debt of the firm if $X_i = hc_n$ for $i = 1, \ldots, n-1$ and $X_n = 1 + hc_n$ where h is defined in (18), as the time period length:

$$\begin{aligned} V_{\mathrm{EM}}(0,T_n) &= \sum\nolimits_{i=1}^{n} D(0,T_i) \\ &= A(0)[1-\mathrm{N}_n^+] + \sum\nolimits_{i=1}^{n} P(0,T_i)X_i\mathrm{N}_i^- \\ &= R_n(0) + \sum\nolimits_{i=1}^{n} P(0,T_i)Q(0,T_i)hc_n + P(0,T_n)Q(0,T_n) \end{aligned} \tag{19}$$

where the forward survival probability notation $Q(t,T_i)$ replaces multi-variate normal probabilities N_i^- Equation (19) resembles (8) remarkably. The only difference is the recovery assumption. Note that the first term $1-\mathrm{N}_n^+$ represents the total unconditional default probability (under a different measure).

[12] Equations (17) and (18) are consistent with Vasicek (1977), but not Cox, Ingersoll, and Ross (1985).

The extended Merton model is usually categorized as a structural model, because of its endogenous recovery assumption and the use of asset value. However, default is defined as the asset value crossing an external barrier. This could cause negative equity value. To see that, we can look at the payoffs at time T_{n-1}:

$$
A(T_{n-1}) > X_{n-1} \quad \Rightarrow \begin{cases} D(T_{n-1}, T_{n-1}) = X_{n-1} \\ D(T_{n-1}, T_n) = P(T_{n-1}, T_n) E^{\mathsf{F}}_{n-1}[\min\{A(T_n), X_n\}] \\ E(T_{n-1}) = P(T_{n-1}, T_n) E^{\mathsf{F}}_{n-1}[\max\{A(T_n) - X_n, 0\}] - X_{n-1} \end{cases}
$$

$$
A(T_{n-1}) \le X_{n-1} \quad \Rightarrow \begin{cases} D(T_{n-1}, T_{n-1}) = A(T_{n-1}) \\ D(T_{n-1}, T_n) = 0 \\ E(T_{n-1}) = 0 \end{cases}
$$

When $A(T_{n-1})$ is small, there is no guarantee that the equity value, $E(T_{n-1})$, can exceed X_{n-1} since X_{n-1} can be arbitrary. To keep the continuity assumption of the asset value at time T_{n-1}, we need to issue new equity when it is negative. In other words, we allow the company to raise new equity when it is already in bankruptcy. Clearly this is not possible in reality. There are three approaches to avoid such a problem. The first is to set the default boundary not for asset, but for equity. That is, let default be the equity value less than the coupon payment, i.e. $E(T_{n-1}) < X_{n-1}$, instead of asset value less than the coupon payment. This is the Geske-Johnson model that we will discuss in the next sub-section. Second, we should treat the underlying asset as an unobservable state variable and specify recover value separately (e.g. Longstaff and Schwartz (1995) and Zhou (2001)). But doing so effectively transforms the structural model into reduced form in that both defaults and recovery are exogenously specified. It differs only slightly from the reduce form approach by different default processes, one assumes a Poisson process and the other assumes a diffusion variable crossing a barrier. Third, we can simplify the debt structure so that an endogenous barrier can be solved (e.g. Leland (1994) and Leland and Toft (1996)).

D. The Geske-Johnson Model

The most direct extension of the Black-Scholes-Merton model is Geske's compound option model (1977). The compound option model provides an exact match between a compound option (option on option) and the equity value of a company with multiple debts. For a company with multiple debts, the survival of the company represents a series of nested call options, identical to a compound option. As a result, the true structural model of Geske (1977) and Geske and Johnson (1984) need to solve for internal strikes.

We extend the Geske-Johnson model of an n period risky debt to incorporate random interest rates.[13, 14] The T_n -maturity zero coupon bond can be written as follows.

$$
\begin{aligned}
(20)\qquad D(0,T_n) = \sum_{i=1}^{n} P(0,T_i)X_i[\Pi_i^-(K_{1n}, \ldots, K_{in}) - \Pi_i^-(K_{1n-1}, \ldots, K_{in-1})] \\
+ A(0)[\Pi_{n-1}^+(K_{1n-1}, \ldots, K_{n-1n-1}) - \Pi_n^+(K_{1n}, \ldots, K_{nn})]
\end{aligned}
$$

where[15]

$$
\Pi_i^{\pm}(K_{1j}, K_{2j}, \ldots, K_{ij}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} N_i(h_1^{\pm}(K_{1j}(r)), h_2^{\pm}(K_{2j}(r)), \ldots, h_i^{\pm}(K_{ij}(r)))\varphi(r(T_1), r(T_2), \ldots, r(T_i)) \, dr(T_1)dr(T_2), \ldots, dr(T_i)
$$

for $n \geq j \geq i$ where φ is the joint density function of various interest rate levels observed at different times under the forward measure. Note that $h_j^{\pm}(K_{ij})$, for $i < j$, is to plug into (18) K_{ij} for the strike and K_{ij} is the internal solution to:[16]

$$
(21)\qquad E(T_i) = X_i
$$

which is a function of the interest rate at time T_i , and $K_{ii} = X_i$. All strikes are solved internally. Note that since interest rates are random, the internally solved strike, K_{ij} , is a function of r under the forward measure. If interest rates are deterministic, then $\Pi_i^{\pm} = N_i(\ )$ the standard multi-variate normal probability function. If we assume that default points are equal to cash obligations, i.e. $K_{ij} = X_i$ for all j, then $\Pi_i^{\pm} = \mathrm{N}_i^{\pm}$ defined in (18).

Although each bond is computed by a complicated formula, the coupon bond is not:

[13] The formulas provided by Geske (1977) are incorrect and corrected by Geske and Johnson (1984). However, Geske and Johnson only present formulas for n = 2. Here, we generalize their formulas to an arbitrary n.

[14] Later on, for calibration, we also augment the model to include non-constant volatility.

[15] Note that the implementation of (20) does not need multi-variate integrals. The easiest way to implement it is to construct a bi-variate lattice. Eom, Helweg, and Huang (2002), for example, use a one-dimension binomial model to implement the deterministic Geske-Johnson model.

[16] Or alternatively, it can be written as:

$$
A(T_i) = \sum_{k=i}^{j} D(T_i, T_k) + X_i
$$

$$V_{\text{GJ}}(0,T_n)=\sum_{i=1}^{n} D(0,T_i)$$

(22)
$$= A(0)[1-\Pi_n^+(K_{1n},K_{2n}, \quad ,K_{nn})]+\sum_{i=1}^{n} P(0,T_i)X_i\Pi_i^-(K_{1n}, \quad ,K_{in})$$
$$= R_n(0)+\sum_{i=1}^{n} P(0,T_i)Q(0,T_i)hc_n + P(0,T_n)Q(0,T_n)$$

where $X_i = hc_n$ for $i < n$ and $X_i = 1+c_n h$ for $i = n$. This equation is extremely similar to (19) except that the probabilities are defined differently due to different strikes. Hence, by observation, the recovery must be $R_n(0) = A(0)[1-\Pi_n^+(K_{1n},K_{2n}, \quad ,K_{nn})]$. We should note that the Geske-Johnson model satisfies the condition $Q(0,T_i) > Q(0,T_{i+1})$

Note again that $\Pi_n^-(K_{1n},K_{2n}, \quad ,K_{nn})$ is the total survival probability (under the forward measure) because the asset value, $A(t)$ needs to stay above its default boundaries, K_{ij} at all times. Hence, the total (cumulative) default probability is $[1-\Pi_n^-(K_{1n},K_{2n}, \quad ,K_{nn})]$. To see it, we know that the total default probability can be derived directly from summing the default probability of each period:

(23)
$$\int_0^{K_{1n}} \varphi(A_1)dA_1 + \int_{K_{1n}}^{\infty}\int_0^{K_{2n}} \varphi(A_1,A_2)dA_2dA_1 + \quad + \int_{K_{1n}}^{\infty} \quad \int_{K_{n-1n}}^{\infty}\int_0^{K_{nn}} \varphi(A_1, \quad ,A_n)dA_n \quad dA_1$$
$$=1-\int_{K_{1n}}^{\infty} \quad \int_{K_{n-1n}}^{\infty}\int_{K_{nn}}^{\infty} \varphi(A_1, \quad ,A_n)dA_n \quad dA_1$$
$$=1-\Pi_n^-(K_{1n}, \quad K_{nn})$$

where $A_i = A(T_i)$ is a short-hand notation. This is analogous to $\int_0^T -dQ(t) = 1-Q(T)$. The recovery is to consider the cash amount received upon default:

(24)
$$P(0,T_1)\int_0^{K_{1n}} A_1\varphi(A_1)dA_1 + P(0,T_2)\int_{K_{1n}}^{\infty}\int_0^{K_{2n}} A_2\varphi(A_1,A_2)dA_2dA_1 +$$
$$+P(0,T_n)\int_{K_{1n}}^{\infty} \quad \int_{K_{n-1n}}^{\infty}\int_0^{K_{nn}} A_n\varphi(A_1, \quad ,A_n)dA_n \quad dA_1$$
$$= A_0\left[1-\int_{K_{1n}}^{\infty} \quad \int_{K_{n-1n}}^{\infty}\int_{K_{nn}}^{\infty} \varphi^+(A_1, \quad ,A_n)dA_n \quad dA_1\right]$$
$$= A_0[1-\Pi_n^+(K_{1n}, \quad K_{nn})]$$

The change of measure can be found in an appendix.

In this unified framework, we can see that the difference in the Geske-Johnson model differs from Jarrow-Turnbull model is that Jarrow and Turnbull assume a fixed recovery value at maturity while Geske and Johnson assume a fixed recovery value at current time.

4. CALIBRATION AND MODEL COMPARISON

In this section, we demonstrate how different models can calibrate to the same market data and imply different parameter values. Equation (8) represents a general formula for all risky bond pricing models, reduced form and structural models included. It is seen that the difference only lies in the recovery assumption: recovery of the face value yields the Jarrow-Turnbull model; recovery of the market value yields the Duffie-Singleton model, and recovery of the asset value yields the extended Merton and the Geske-Johnson model.

We shall first use a two-period model as an example to demonstrate the computational details. Then we provide analysis in a multi-period setting. The Vasicek model (1977) for the risk free term structure of interest rates is considered.

A. A Two-Period Example

We need three pieces of information to complete the calibration: risk free zero yield curve, a set of risky bond prices, and a recovery assumption. In the case of structural models, the recovery assumption is replaced by the volatility curve for the set of bond prices.

To demonstrate the calibration procedures for both reduced form and structural models, we use a two-period example. The following table describes the base case:

time	1	2
coupon	10	10
face	100	100
bond price	100	100
yield curve	5%	5%

The Jarrow-Turnbull model can be calibrated as follows. From (11), we obtain the one year bond formula:

$$V_{\mathrm{JT}}(0,1) = P(0,1)[Q(0,1)(1+c_1)+(1-Q(0,1))w_1]$$

We assume a fixed recovery rate of 0.4 of the principal and accrued interest. In this one-year example, the recovery amount is \$44 if default occurs. Given coupon $c_1 = \$10/\$100 = 10\%$ and one-year discount $P(0,1) = e^{-5\%} = 0.9512$, we can solve for the survival probability to be $Q(0,1) = 0.9262$. Again, from (11), for the two-year bond:

$$V_{\mathrm{JT}}(0,2) = P(0,1)Q(0,1)c_2 + P(0,2)Q(0,2)(1+c_2) + P(0,1)[1-Q(0,1)]w_2 + P(0,2)[Q(0,1)-Q(0,2)]w_2$$

With the knowledge of $Q(0,1)$, together with $P(0,2) = e^{-10\%} = 0.9048$, we can then solve for the second period survival probability to be $Q(0,2) = 0.8578$.

The Duffie-Singleton model can be calibrated as follows. From (13), we have:

$$V_{\mathrm{DS}}(0,1) = P(0,1)Q^{*}(0,1)(1+c_1)$$

for the one-year bond and

$$V_{\mathrm{DS}}(0,2) = P(0,1)Q^{*}(0,1)c_2 + P(0,2)Q^{*}(0,2)(1+c_2)$$

for the two-year bond. Hence, we solve for $Q^{*}(0,1) = 0.9557$ and $Q^{*}(0,2) = 0.9134$. From (14), we know that $Q^{*}(0,t) > Q(0,t)$ always due to non-negative recovery. By assuming a recovery rate of 0.4, we get $Q(0,1) = 0.8929$ and $Q(0,2) = 0.7973$, which are both lower than the survival probabilities calculated by the Jarrow-Turnbull model. Note that in both models, the recovery present values for one and two-year bonds are 3.09 and 5.81 for Jarrow-Turnbull and 6.57 and 12.15 for Duffie-Singleton, respectively. It is seen that the recovery amounts for Duffie-Singleton are higher, and hence to maintain the same the bond values, the survival (default) probabilities of Duffie-Singleton have to be lower (higher) to balance out.

In both the Jarrow-Turnbull model and the Duffie-Singleton model, given that there is no other random factor other than the interest rate, there is no need to identify a specific term structure model, given that survival and default probabilities are computed under the forward measure. However, in the Geske-Johnson model, in addition to random interest rates, there is a random "asset price" state variable. As a result, a specific term structure model needs to be specified in order to carry out survival and default probabilities.

To simplify the calculation and without any loss of generality, we assume a deterministic yield curve in this sub-section. The Vasicek term structure model is assumed in the next sub-section. In the Geske-Johnson model, the recovery amount is random and endogenous. Under deterministic interest rates, the one-period Geske-Johnson model is a Black-Scholes-Merton solution:

$$V_{\mathrm{GJ}}(0,1) = D(0,1) = P(0,1)X_1 N(h_1^{-}(X_1)) + A(0)[1 - N(h_1^{+}(X_1))] \tag{25}$$

where $h_1^{\pm}$ is defined in (18). Since interest rates are non-stochastic, $P(0,1) = \Lambda(0,1)$. Equation (25) is identical to (18) when $\sigma_P = 0$ and $T_n = T_1 = 1$. For the two-year bond, i.e. $T_n = T_2 = 2$ and $T_1 = 1$, we have two zero bond components:

$$D(0,1) = P(0,1)X_1 N(h_1^{-}(X_1)) + A(0)[1 - N(h_1^{+}(X_1))] \tag{26}$$

and

$$D(0,2) = P(0,1)X_1[N_1(h_1^-(K_{12})) - N_1(h_1^-(X_1))] + P(0,2)X_2N_2(h_1^-(K_{12}), h_2^-(X_2); \sqrt{1/2}) + A(0)[N_1(h_1^+(X_1)) - N_2(h_1^+(K_{12}), h_2^+(X_2); \sqrt{1/2})] \tag{27}$$

where $h_i^\pm$ for i = 1, 2 is defined in (18), $X_1 = c_2$, $X_2 = 1 + c_2$. Recall that K_{12} is the solution to $A(1) = D(1,2) + X_1$ where $D(1,2)$ is a bond price at $t = 1$. Finally, the two-year coupon bond value is the sum of(26) and (27):

$$\begin{aligned} V_{\mathrm{GJ}}(0,2) &= D(0,1) + D(0,2) \\ &= P(0,1)X_1N(h_1^-(X_1)) + P(0,2)X_2N_2(h_1^-(K_{12}), h_2^-(X_2)) \\ &\quad + A_0[1 - N_2(h_1^+(K_{12}), h_2^+(X_2); \sqrt{1/2})] \end{aligned} \tag{28}$$

In the equation, we need to evaluate three probabilities. The first one is $N(h_1^-)$ which has already been evaluated. The second one is a bivariate normal probability with two separate strikes, K_{12} that has to be internally solved and X_2 that is the face and coupon value at maturity.

Note that X_1 is the first cash flow of the company, hence $X_1 = 10 + 110 = 120$. The value of $D(0,1)$ hence contains the cash flow of the first bond and the coupon amount of the second bond. Here, we assume that the split of $D(0,1)$ is proportional. That is the value of the one-year bond is $\frac{110}{120}D(0,1)$, and the value of the two-year bond is $\frac{10}{120}D(0,1) + D(0,2)$.

In the original Geske-Johnson model, the volatility is assumed flat (as in Black-Scholes (1973)). Unfortunately under this condition, the calibration of the second bond becomes impossible. Hence, we extend the model to include a volatility curve, i.e. $v(0,2)^2 = v(0,1)^2 + v(1,2)^2$. This flexibility allows us to calibrate the model to the two-year bond price. The results are $A(0) = 2351.65$, $v(0,1) = 1.5$, and $v(1,2) = 0.69$. Under such results, the survival probabilities are $Q(0,1) = N_1^-(K_{12}) = 0.9426$ and $Q(0,2) = N_2^-(K_{12}, X_2) = 0.8592$ respectively. And the recovery values for the two bonds are computed by subtracting the coupon value from the corresponding discount debt value. For example, $D(0,1) = 109.09$ which is split into two parts – the one-year bond of \$100 and the coupon of the two-year bond of \$9.09. The one-year bond has a coupon portion of $110 \times Q(0,1) \times P(0,1) = 110 \times 0.9426 \times 0.9512 = 98.63$, and hence has a recovery value of $100 - 98.63 = 1.37$. A similar calculation gives the recovery value of the two-year bond as 5.52.[17]

For easy comparison, we put together all the numbers in the following table. We observe that the Duffie-Singleton (DS) model has the lowest survival probabilities and therefore should have the highest recovery values, due to the fact that they both contribute positively to the bond price. The Geske-Johnson (GJ) model has the highest survival probabilities and lowest recovery values. The Jarrow-Turnbull (JT)

[17] Note that the total recovery value needs to be 6.89 by (28).

model is in between. Since the bond price is traded at par, higher survival probabilities need to be balanced by lower recovery values.

	JT	DS	GJ
$Q(0,1)$	0.9262	0.8929	0.9426
$Q(0,2)$	0.8578	0.7923	0.8592
total recovery value (i.e. $R(0)$)	8.90	18.68	6.89
recovery of first bond	3.09	6.53	1.37
recovery of second bond	5.81	12.15	5.52

B. Multi-period Analysis

In this section, we examine the multi-period behavior of reduced form models, namely Jarrow-Turnbull and Duffie-Singleton and the structural model of Geske-Johnson. We examine the default and recovery implication under an *n*-period setting. We also incorporate stochastic interest rates in our analysis. We replace the interest rate process in (16) by the following Ornstein-Uhlenbeck process:

(29) $$dr(t) = \alpha\left(\mu - \frac{\delta q}{\alpha} - r(t)\right)dt + \delta dW_r^Q(t)$$

where the parameter values are given for flat (constant) and upward sloping yield curves as follows

Vasicek Model Parameters	
α (reverting speed)	0.40
μ (reverting level)	0.08
δ (volatility)	0.05
q (mkt. price of risk)	-0.10
$r(0)$ (initial rate)	0.05

We first examine flat yield curve. Then we use the upward sloping parameter values to construct the Vasicek model.

It is generally thought that the Geske-Johnson model is difficult to implement because for an *n*-period bond, we need *n*-dimensional probability functions, which are computationally expensive. However, in this paper, we employ the standard one-dimensional equity binomial model, which can be accurate to the second decimal place in 50 steps.

We first examine the case of extremely low coupons. This is the case where we can see the fundamental difference between the structural model of Geske-Johnson and the reduced form models of Jarrow-Turnbull and Duffie-Singleton. Note that the Jarrow-Turnbull model is (11) and the Duffie-Singleton model is (13) and (14), both with $c_n = 0$. The Geske-Johnson model is (22) with $X_i = 0$ for $i < n$. In all the models, the zero coupon bond price, $P(t,T)$, should follow the Vasicek model (formula given in an appendix.) The face value of debt is 110. We run the Geske-Johnson model with an asset value of $A(0) = 184$ and various volatility levels: 0.4, 0.6, 1.0, and 1.6. The results computed are summarized as follows.

GJ Model				
volatility	0.4	0.6	1.0	1.6
equity value	169.25	177.92	183.49	183.94
debt value	14.74	6.08	0.51	0.06
recovery	4.06	2.08	0.19	0.00
JT Model				
recovery rate	3.69%	1.89%	0.17%	0.00%
intensity	1.79%	4.85%	13.19%	20.00%

As the volatility goes up, the equity value in the Geske-Johnson model goes up (i.e. call option value goes up.) Since the asset value is fixed at 184, the debt value goes down. The survival probability curves under various volatility scenarios of the Geske-Johnson model are plotted in Figure 1; and the default probability curves (unconditional, i.e. $Q(T_{i-1}) - Q(T_i)$) are plotted in Figure 1a. We observe several results. First, as the risk of default becomes eminent (i.e. high volatility and low debt value), the likelihood of default shifts from far term (peak at year 30 for volatility = 0.4) to near term (peak at year 5 for volatility = 1.6). Second, it is seen that the asset volatility has a huge impact on the shape of the survival probability curve. The Geske-Johnson model is able to generate humped default probability curve, often observed empirically. Third, these differently shaped probability curves are generated by one single debt, something not possible in reduced form models. Both the Jarrow-Turnbull and Duffie-Single models cannot generate such probability curves with one single bond, due to the lack of information of intermediate cash flows. Corresponding to the recovery amounts under the Geske-Johnson model, we set the fixed recovery rate of the Jarrow-Turnbull model as shown in the above table. Given that one bond can only imply one intensity parameter value, we set it (under each scenario) so that the zero coupon bond price generated by the Geske-Johnson model is matched. As shown in the above table, the intensity rate goes from 1.79% per annum to 20% per annum. Note that flat intensity value is equivalent to a flat *conditional* default probability curve. To visualize the difference this flat conditional default probability curve of the Jarrow-Turnbull model with the non-flat curve generated by the Geske-Johnson model, we plot in Figure 1c the case of volatility = 1.6. The conditional default probabilities are calculated as $\frac{Q(T_{i-1})-Q(T_i)}{Q(T_{i-1})}$.

Our next analysis is to keep bond value fixed, so that we can examine the default probability curve with the risk of the bond controlled. We assume that the company issues only one coupon bond at 10%. At the volatility level of 0.4, the asset value is \$123, at 0.6, it is \$290, at 1.0, it is \$15,000. Figure 2 and Figure 2a demonstrate the Geske-Johnson model for various volatility levels but keep the bond at par. We can see that for the same par bond, the default and survival probability curves are drastically different as the asset/volatility combination changes. This is a feature not captured by either the Duffie-Singleton or the Jarrow-Turnbull model.

To compare to the Jarrow-Turnbull and Duffie-Turnbull models, we keep the case where the volatility level is 0.6 and asset value is 290. The recover rates of both Jarrow-Turnbull and Duffie-Singleton models are assumed to be 0.4. Figure 3 and Figure 3a show the survival and default probability curves of the three models. The flat conditional forward default probability for the Jarrow-Turnbull model is solved to be 7.60% (or equivalently the intensity rate is 7.74%.) The conditional forward default probabilities for the Duffie-Singleton model are certainly non-constant. The "recovery-adjusted" continuously compounded discount rate is 9.52%. From the survival probability curves (Figure 3), it is seen that the Geske-Johnson and Jarrow-Turnbull models can be close. But the default probability curves (Figure 3a) demonstrate that the default pattern can be quite different.

5. CREDIT DEFAULT SWAP PRICING

Credit default swaps are the most widely traded credit derivative contract today. A default swap contract offers protection against default of a pre-specified corporate issue. In the event of default, a default swap will pay the principal (with or without accrued interest) in exchange for the defaulted bond.[18]

Default swaps, like any other swap, have two legs. The premium leg contains a stream of payments, called spreads, paid by the buyer of the default swap to the seller till either default or maturity, whichever is earlier. The other leg, protection leg, contains a single payment from the seller to the buyer upon default if default occurs and 0 if default does not occur. Under some restrictive conditions, credit default swap spreads are substitutes for par floater spreads.[19] In many occasions, the traded spreads off credit default swaps are more representative than those off risky corporate bonds. The valuation of a credit default swap is straightforward. For the default protection leg:

(30)
$$\begin{aligned} W(0,T_n) &= \int_0^{T_n} (1-w(A(t),r(t)))P(0,t)[-dQ(0,t)] \\ &= \int_0^{T_n} P(0,t)[-dQ(0,t)] - \int_0^{T_n} w(A(t),r(t))P(0,t)[-dQ(0,t)] \\ &= \int_0^{T_n} P(0,t)[-dQ(0,t)] - R_n(0) \end{aligned}$$

---

[18] Default swaps can also be designed to protect a corporate name. These default swaps were used to be digital default swaps. Recently these default swaps have a collection of "representative" reference bonds issued by the corporate name. Any bond in the reference basket can be used for delivery.

[19] See, for example, Chen and Soprazetti (2002) for a discussion of the relationship between the credit default swap spread and the par floater spread.

In discrete time, we can write (30) as:

(30a) $$W(0,T_n) = \sum_{i=1}^{n} P(0,T_i)[Q(0,T_{i-1}) - Q(0,T_i)] - R_n(0)$$

This is called the protection leg or the floating leg of the swap. For the premium leg, or the fixed leg:

(31) $$W(0,T_n) = s_n \sum_{i=1}^{n} P(0,T_i)Q(0,T_i)$$

Combining (30) and (31), we can use market credit default swap spreads to back out default probability curve. As the default swap market grows, more and more investors seek arbitrage trading opportunities between corporate bonds and default swaps. This suggests that we should use the calibrated corporate bond curves (last section) to compute default swap spreads. We use the results of Figure 3 to compute credit default swap values for various tenors (1~30 years). The recovery rate in the swap contract is assumed to be 0.[20] Figure 4 shows how different models can imply different default swap values.[21] The Geske-Johnson model are close to the Duffie-Singleton model at near terms but close to the Jarrow-Turnbull model at far terms. As we have seen, same par bond implies very different default probabilities from the Jarrow-Turnbull, Duffie-Singleton, and Geske-Johnson models, which in turn imply very different credit default swap values.

The default swap market has grown very fast in the past few years[22] and many fixed income traders and fund managers try to arbitrage between corporate bonds and credit default swaps should they see discrepancies in spreads. Here, we demonstrate that such arbitrage trading strategies can be misleading. Arbitrage profits can be entirely due to model specification. To see that, we suppose that the Geske-Johnson model is the correct model. Hence the probabilities that calculate the par bond are shown in Figure 5 for the case where asset value is $290 and volatility is 0.6 (so that the coupon debt is at par). The 30-year default swap spread implied by the Geske-Johnson model is 438 basis points. This is done by implementing (30a) to obtain the default swap value ($32.87) and implementing (31) to solve for the spread. Note that the credit default swap value and spread computed using the Geske-Johnson model are consistent with the recovery assumption of the Geske-Johnson model. To use the Jarrow-Turnbull model, we need a fixed recovery rate. To get such value, we use that the probabilities generated by the Geske-

[20] As long as the recovery rate is fixed, it simply scales up/down the curves and does not change the shapes.
[21] We should note that the default swap price cannot be computed directly from the Duffie-Singleton model because there is no cash flow paid if there is no default on the protection leg. Hence, for the Duffie-Singleton model, it is possible to compute the default swap value once the underlying bond is available; but not possible to compute the bond price when the default swap spread is available.
[22] According to a Lehman Brothers credit research report (O'kane (2001)), the credit derivative markets are estimated to be $1 trillion in notional at the time the report was written and near half of which is the market of default swaps.

Johnson model and a fixed recovery rate to compute the default swap value. This implied recovery rate is 0.5840. Now we let the Jarrow-Turnbull model to calibrate to the data (by changing the intensity paramter). The Jarrow-Turnbull bond price is it is 110.10 at an intensity level of 7.19%, 10% overvalued due to model error.

The situation can be even more severe if we allow the Geske-Johnson model to generate more humped shaped default probability curve. Set volatility to 0.4 and asset value to 184, we price another 30-year par bond by the Geske-Johnson model. We also use the Vasicek model for the term structure. Again, assume the Geske-Johnson model to be correct. It implies the probability curves as shown in Figure 6. The 30-year default swap value is $11.18 and the spread is 163 basis points. A flat recovery rate implied by such a spread is 0.3117. Using this recovery rate, we obtain the Jarrow-Turnbull price to be 83.74 at the intensity level of 4.99%, which is 16% undervalued due to model error. If we set volatility to 0.6 and asset value to 184, we price the 30-year bond by the Geske-Johnson model at 80. Again, assume the model to be correct. It implies the probability curves as shown in Figure 7. The 30-year default swap value is $29.06 and the spread is 562 basis points. A flat recovery rate implied by such a spread is 0.5485. Using this recovery rate, we obtain the Jarrow-Turnbull price to be 81.10 at the intensity level of 9.84%, 1% overvalued due to model error.

6. CONCLUSION

In this paper, we provide a general framework that brings consistency between the reduced form and structural models. The structural models we consider in this paper is not those of the "barrier-type" that assumes exogenous barriers but the Geske-Johnson model that allows the default points (strike price) to be endogenously computed. We show that the "true structural" model of Geske-Johnson can be simplified to barrier-type (extended Merton) if the endogenous default points are not internally solved for but exogenously given.

In a discrete time binomial framework, we show that any set of risky cash flows and recovery can be priced by a simple formula. This formula is same for both structural and reduced form models. This formula allows us to compare various models because they only differ in the recovery computation. Different recovery assumptions result in different survival and default probabilities. In calibration, the differences in recovery amounts and in probabilities balance each other out, as we demonstrate, because each model price is made to match the market price. However, these different recovery values and probabilities should result in large differences in derivatives prices. We demonstrate that large differences exist even for the simplest credit derivative contract – credit default swaps.

Finally, In order to compare the "true structural" Geske-Johnson model with the reduced form models represented by Jarrow and Turnbull (1995) and Duffie and Singleton (1997, 1999), we extend the Geske-Johnson model to incorporate random interest rates and a non-flat volatility curve. In a series of appendices, we demonstrate how to implement the Geske-Johnson model with random interest rates and a volatility curve.

APPENDIX

A. Derivation of (18)

We shall derive (18) by induction. A T_1-maturity zero coupon bond with a barrier X_1 is:

$$\begin{aligned} D(0,T_1) &= E_0^{\mathsf{Q}}\left[e^{-\int_0^{T_1} r(u)du} \min\{A(T_1), X_1\}\right] \\ &= P(0,T_1)E_0^{\mathsf{F}_1}\left[\min\{A(T_1), X_1\}\right] \\ &= A(0)[1-N(h_1^+(X_1))] + P(0,T_1)X_1 N(h_1^-(X_1)) \end{aligned} \tag{A1}$$

where $h_1^{\pm}(X_1)$ is defined in the text. A T_2-maturity zero coupon bond with a barrier X_2 has the following value at T_1:

$$\begin{cases} E_{T_1}^{\mathsf{Q}}\left[e^{-\int_{T_1}^{T_2} r(u)du} \min\{A(T_2), X_2\}\right] & A(T_1) > X_1 \\ 0 & A(T_1) \le X_1 \end{cases} \tag{A2}$$

Hence, it has the following value today:

$$\begin{aligned} D(0,T_2) &= E_0^{\mathsf{Q}}\left[e^{-\int_0^{T_1} r(u)du} E_{T_1}^{\mathsf{Q}}\left[e^{-\int_{T_1}^{T_2} r(u)du} \min\{A(T_2), X_2\}\right] I_{\{A(T_1)>X_1\}} \right] \\ &= E_0^{\mathsf{Q}}\left[E_{T_1}^{\mathsf{Q}}\left[e^{-\int_0^{T_2} r(u)du} \left(A(T_2) I_{\{A(T_2)<X_2\}} + X_2 I_{\{A(T_2)>X_2\}}\right)\right] I_{\{A(T_1)>X_1\}} \right] \\ &= P(0,T_2)E_0^{\mathsf{F}_n}\left[A(T_2) I_{\{A(T_2)<X_2 \wedge A(T_1)>X_1\}} + X_2 I_{\{A(T_2)>X_2 \wedge A(T_1)>X_1\}}\right] \\ &= P(0,T_2)E_0^{\mathsf{F}_n}\left[A(T_2)\left(I_{\{A(T_1)>X_1\}} - I_{\{A(T_2)>X_2 \wedge A(T_1)>X_1\}}\right) + X_2 I_{\{A(T_2)>X_2 \wedge A(T_1)>X_1\}}\right] \\ &= A(0)[\mathsf{N}_1^+ - \mathsf{N}_2^+] + P(0,2)X_2 \mathsf{N}_2^- \end{aligned} \tag{A3}$$

where the second to last line is obtained by appropriately dividing the integration region. Carrying out the expectation using the standard log normal results should yield the result desired. Similar procedure is applied to any arbitrary T_n.

B. N_n^+ *and* N_n^-

In this appendix, we show that N_n^+ and N_n^- are both survival probabilities, but under different probability measures. For the ease of composition, we use $\mathsf{N}_1^{\pm} = N(h_1^{\pm}(X_1))$ where $N(\cdot)$ is a standard univariate normal probability function, as an example. The general case, while tedious algebraically, is straightforward.

To simplify notation, we drop “1” from $h_1^{\pm}$, X_1, T_1, and measure F_1. Note that:

(B1)
$$
\begin{aligned}
N(h^+) &= \frac{P(0,T)}{A(0)} E_0^{\mathsf{F}}[A(T) I_{\{A(T)>X\}}] \\
N(h^-) &= E_0^{\mathsf{F}}[I_{\{A(T)>X\}}]
\end{aligned}
$$

Equation (B1) can be written as:

(B2)
$$
\begin{aligned}
E_0^{\mathsf{F}}[A(T) I_{\{A(T)>X\}}] &= E_0^{\mathsf{F}}[A(T)] E_0^{\mathsf{F}^*}[I_{\{A(T)>X\}}] \\
&= \frac{A(0)}{P(0,T)} E_0^{\mathsf{F}^*}[I_{\{A(T)>X\}}]
\end{aligned}
$$

The change of measure from F to F^* is done as follows:

(B3) $$\frac{d\mathsf{F}^*}{d\mathsf{F}} = \frac{A(T)}{E_0^{\mathsf{F}}[A(T)]}$$

Given that A is a log normal process, we immediately obtain that the above derivative as a change of measure of $\sqrt{\mathrm{var}[\ln(A(T)]}$ under the F measure. In the following, we sketch the basic math of forward measure.

Assume an interest rate process under the Q measure generally as:

(B4) $$dr(t) = \alpha(r,t)dt + \sigma(r,t)dW^{\mathsf{Q}}(t)$$

Then, by (3), we have the Radon-Nykodym derivative defined as follows:

(B5) $$\eta(t,T) = \frac{\Lambda(t,T)}{P(t,T)}$$

Applying Ito’s lemma on the bond price:

(B6)

$$0=\ln P(T,T)=\ln P(t,T)+\int_t^T \frac{1}{P(u,T)}\left[P_u(u,T)du+P_r(u,T)dr+\frac{1}{2}P_{rr}(u,T)(dr)^2\right]dW^{Q}(u)$$

$$-\int_t^T \frac{1}{2}\left[\frac{\sigma(r,u)P_r(u,T)}{P(u,T)}\right]^2 du$$

$$=\ln P(t,T)+\int_t^T \frac{1}{P(u,T)}\left[P_u(u,T)du+P_r(u,T)\hat{\mu}(r,u)+\frac{1}{2}P_{rr}(u,T)\sigma(r,u)^2\right]du$$

$$+\int_t^T \frac{1}{P(u,T)}P_r(u,T)\sigma(r,u)dW^{Q}(u)-\int_t^T \frac{1}{2}\left[\frac{\sigma(r,u)P_r(u,T)}{P(u,T)}\right]^2 du$$

$$=\ln P(t,T)+\int_t^T r(u)du+\int_t^T \frac{1}{P(u,T)}P_r(u,T)\sigma(r,u)dW^{Q}(u)-\int_t^T \frac{1}{2}\left[\frac{\sigma(r,u)P_r(u,T)}{P(u,T)}\right]^2 du$$

Letting:

(B7) $$\theta(t,T)=\frac{\sigma(r,t)P_r(t,T)}{P(t,T)}$$

and moving the first two terms to the left:

(B8)
$$-\int_t^T r(u)du-\ln P(t,T)=\int_t^T \theta(u,T)dW^{Q}(u)-\int_t^T \frac{1}{2}\theta(u,T)^2 du$$

$$\frac{\Lambda(t,T)}{P(t,T)}=\eta(t,T)=\exp\left(\int_t^T \theta(u,T)dW^{Q}(u)-\int_t^T \frac{1}{2}\theta(u,T)^2 du\right)$$

This implies the Girsonav transformation of the following:

(B9) $$W^{Q}(t)=W^{F_T}(t)+\int_t^T \sigma(r,u)\frac{P_r(u,T)}{P(u,T)}du$$

The interest rate process under the forward measure henceforth becomes:

(B10) $$dr(t)=\left[\alpha(r,t)+\sigma(r,t)^2\frac{P_r(t,T)}{P(t,T)}\right]dt+\sigma(r,t)dW^{F_T}(t)$$

Note that the forward measure derived above is quite general. It does not depend on any specific assumption on the interest rate process. In the following, “random equity and random interest rate”, we do need normally distributed interest rates, or there is no solution to option pricing.

$$\text{(B11)} \qquad \begin{bmatrix} \dfrac{dA(t)}{A(t)} \\ dr(t) \end{bmatrix} = \begin{bmatrix} r(t) \\ \alpha(r,t) \end{bmatrix} dt + \begin{bmatrix} \sigma_A & 0 \\ 0 & \sigma_r(r,t) \end{bmatrix} \times \begin{bmatrix} \sqrt{1-\rho^2} & \rho \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} dW_S^{\mathbb{Q}}(t) \\ dW_r^{\mathbb{Q}}(t) \end{bmatrix}$$

Using the forward measure, we get:

$$\text{(B12)} \qquad \begin{bmatrix} \dfrac{dA(t)}{A(t)} \\ dr(t) \end{bmatrix} = \begin{bmatrix} r(t) + \rho\sigma_A\sigma_r(r,t)\dfrac{P_r(t,T)}{P(t,T)} \\ \alpha(r,t) + \sigma_r(r,t)^2 \dfrac{P_r(t,T)}{P(t,T)} \end{bmatrix} dt + \begin{bmatrix} \sigma_A & 0 \\ 0 & \sigma_r(r,t) \end{bmatrix} \times \begin{bmatrix} \sqrt{1-\rho^2} & \rho \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} dW_S^{\mathbb{Q}}(t) \\ dW_r^{\mathbb{F}_T}(t) \end{bmatrix}$$

$$\text{(B13)} \qquad A(T) = A(t)\exp\left(\int_t^T \left[r(u) + \rho\sigma_A\sigma_r(r,t)\frac{P_r(u,T)}{P(u,T)} \right] du + \int_t^T \sigma dW^{\mathbb{F}_T}(u) - \frac{1}{2}\int_t^T \sigma^2 du \right)$$

where

$$dW^{\mathbb{F}_T}(t) = \sqrt{1-\rho^2}\, dW_S^{\mathbb{Q}}(t) + \rho dW_r^{\mathbb{F}_T}(t)$$

From the result of Ito’s lemma, we can then arrive at the following result:

$$\text{(B14)} \qquad \begin{aligned} A(T) = \frac{A(t)}{P(t,T)} \exp\Bigg(\int_t^T \left[-\frac{1}{2}\left[\sigma_r(r,t)\frac{P_r(u,T)}{P(u,T)} \right]^2 + \rho\sigma_A\sigma_r(r,t)\frac{P_r(u,T)}{P(u,T)} - \frac{1}{2}\sigma^2 \right] du \\ - \int_t^T \sigma_r(r,t)\frac{P_r(u,T)}{P(u,T)} dW_r^{\mathbb{F}_T}(u) + \int_t^T \sigma_A dW^{\mathbb{F}_T}(u) \Bigg) \end{aligned}$$

Hence,

$$\text{(B15)} \qquad \begin{aligned} & E_t^{\mathbb{F}_T}[\ln S(T)] = \ln S(t) - \ln P(t,T) + \frac{1}{2}v(t,T)^2 \\ & \mathrm{var}_t^{\mathbb{F}_T}[\ln S(T)] = v(t,T)^2 \end{aligned}$$

where

$$v(t,T)^2 = \int_t^T \left[\left| \sigma_r(r,t) \frac{P_r(u,T)}{P(u,T)} \right|^2 - 2\rho\sigma_A \sigma_r(r,t) \frac{P_r(u,T)}{P(u,T)} + \sigma_A^2 \right] du$$

C. Derivation of (20)

The Geske-Johnson model is usually written as the multi-variate normal probability format. As a result, the final solution looks more complex than it really is. Once, we understand the recursive relationship in zero coupon bond formulas, the final result is very straightforward to recognize. Hence, in this appendix, we provide a simple three period example to demonstrate how we can easily derive and streamline the Geske-Johnson model. The rest can be obtained by induction.

For the one-year zero, it is identical to the Merton-Rabinovitch model:

(C1)
$$\begin{aligned} D(0,T_1) = V_{\mathrm{GJ}}(0,T_1) &= P(0,T_1)K_{11}[N_1(h_1^-(K_{11})) - 0] + A(0)[1 - N_1(h_1^+(K_{11}))] \\ &= P(0,T_1)K_{11}\Pi_1^-(K_{11}) + A(0)[1 - \Pi_1^+(K_{11})] \end{aligned}$$

Note that $K_{11} = X_1$, the first coupon. The second line of the above equation is merely a notation change. The simpler notation allows to more easily label high dimension normal probabilities with different strikes.

For the two-year zero, we need to solve for an internal solution K_{12} that equates

(C2)
$$A(T_1) = D(T_1,T_2) + X_1$$

where

(C3)
$$\begin{aligned} D(T_1,T_2) &= E_{T_1}^{Q}[\Lambda(T_1,T_2)\min\{A(T_2), X_2\}] \\ &= P(T_1,T_2)N(d_-) + A(T_1)[1 - N(d_+)] \end{aligned}$$

and

(C4)
$$d_{\pm} = \frac{\ln A(T_1) - \ln P(T_1,T_2) - \ln X_2 \pm \frac{1}{2} v^2(T_1,T_2)}{v(T_1,T_2)}$$

is a Merton-Rabinovitch result again. Note that $v^2(\cdot,\cdot)$ is defined in (18). We should note that since $P(T_1,T_2)$ is a function of the interest rate at time T_1, i.e. $r(T_1)$. As a result, K_{12} is also a function of $r(T_1)$. For the convenience of later derivation, we rewrite (D3) in its original integral form as follows. Also for notational convenience, we shorten the following notation: $D(T_i,T_j) = D_{ij}$, $P(T_i,T_j) = P_{ij}$, $A(T_i) = A_i$ and $r(T_i) = r_i$.

(C3a) $$D_{12} = \int_{-\infty}^{\infty}\int_{X_2}^{\infty} \Lambda_{12} X_2 \varphi(A_2, r_2) dA_2 dr_2 + \int_{-\infty}^{\infty}\int_{0}^{X_2} \Lambda_{12} A_2 \varphi(A_2, r_2) dA_2 dr_2$$

This expression allows us to easily integrate with other integrals. The value of the two-year zero price at T_1 is:

(C4) $$\begin{cases} D_{12} & A_1 > K_{12} \\ A_1 - X_1 & X_1 < A_1 \le K_{12} \\ 0 & A_1 \le X_1 \end{cases}$$

To obtain the current value of the two-year zero, we simply integrate these payoffs at its corresponding region. Note that K_{12} is a function of the interest rate.

(C5) $$D(0,2) = \underbrace{\int_{-\infty}^{\infty}\int_{K_{12}}^{\infty} \Lambda_{0,1} E[\Lambda_{1,2} \min\{A_2, F_2\}] \varphi(A_1, r_1) dA_1 dr_1}_{\Delta_1} + \underbrace{\int_{-\infty}^{\infty}\int_{X_1}^{K_{12}} \Lambda_{0,1} (A_1 - F_1) \varphi(A_1, r_1) dA_1 dr_1}_{\Delta_2}$$

The second term can be shown as:

(C6) $$\begin{aligned} \Delta_2 &= \int_{-\infty}^{\infty}\int_{K_{11}}^{\infty} \Lambda_{01}(V_1 - F_1)\varphi(V_1, r_1) dV_1 dr_1 - \int_{-\infty}^{\infty}\int_{K_{12}}^{\infty} \Lambda_{01}(V_1 - F_1)\varphi(V_1, r_1) dV_1 dr_1 \\ &= V_0[\Pi_1^+(K_{11}) - \Pi_1^+(K_{12})] - P_{01} X_1 [\Pi_1^-(K_{11}) - \Pi_1^-(K_{12})] \end{aligned}$$

The first term can be valued as:

$$\Delta_1 = \int_{-\infty}^{\infty}\int_{K_{12}}^{\infty} \Lambda_{01}\left[\int_{-\infty}^{\infty}\int_{0}^{K_{22}} \Lambda_{12}A_2\varphi(A_2, r_2 \mid A_1, r_1)dA_2dr_2\right]\varphi(A_1, r_1)dA_1dr_1$$

$$+\int_{-\infty}^{\infty}\int_{K_{12}}^{\infty} \Lambda_{01}\left[\int_{-\infty}^{\infty}\int_{K_{22}}^{\infty} \Lambda_{12}X_2\varphi(A_2, r_2 \mid A_1, r_1)dA_2dr_2\right]\varphi(A_1, r_1)dA_1dr_1$$

(C7)

$$= \int_{-\infty}^{\infty}\int_{K_{12}}^{\infty} \Lambda_{0,1}\left[\int_{-\infty}^{\infty}\int_{0}^{\infty} \Lambda_{12}A_2\varphi(A_2, r_2 \mid A_1, r_1)dA_2dr_2\right]\varphi(A_1, r_1)dA_1dr_1$$

$$-\int_{-\infty}^{\infty}\int_{K_{12}}^{\infty} \Lambda_{01}\left[\int_{-\infty}^{\infty}\int_{K_{22}}^{\infty} \Lambda_{12}A_2\varphi(A_2, r_2 \mid A_1, r_1)dA_2dr_2\right]\varphi(A_1, r_1)dA_1dr_1$$

$$+\int_{-\infty}^{\infty}\int_{K_{12}}^{\infty} \Lambda_{01}\left[\int_{-\infty}^{\infty}\int_{K_{22}}^{\infty} \Lambda_{12}X_2\varphi(A_2, r_2 \mid A_1, r_1)dA_2dr_2\right]\varphi(A_1, r_1)dA_1dr_1$$

$$= A_0\Pi_1^+(K_{12}) - A_0\Pi_2^+(K_{12}, K_{22}) + P_{02}X_2\Pi_2^-(K_{12}, K_{22})$$

Hence, the two-year coupon bond, returning to the original notation, is:

(C8)

$$\begin{aligned} V_{\mathrm{GJ}}(0,T_2) &= D(0,T_1) + D(0,T_2) \\ &= D(0,T_1) + \Delta_1 + \Delta_2 \\ &= P(0,T_1)X_1\Pi_1^-(X_1) + P(0,T_2)X_2\Pi_2^-(K_{12}, X_2) + A_0[1 - \Pi_2^+(K_{12}, X_2)] \end{aligned}$$

where

(C9)

$$\Pi_1^{\pm}(K) = \int_{-\infty}^{\infty}\int_{K}^{\infty} \varphi^{\pm}(A_1, r_1)dA_1dr_1$$

$$\Pi_2^{\pm}(K_1, K_2) = \int_{-\infty}^{\infty}\int_{K_1}^{\infty}\int_{K_2}^{\infty} \varphi^{\pm}(A_1, A_2, r_1)dA_2dA_1dr_1$$

It is shown in Appendix A that the last two integrals in the above equation can be written as:

(C10) $$\int_{K}^{\infty} A_1\varphi^-(A_1 \mid r_1)dA_1 = A_0\int_{K}^{\infty} \varphi^+(A_1 \mid r_1)dA_1$$

and

(C11) $$\int_{K_1}^{\infty}\int_{K_2}^{\infty} A_2\varphi^-(A_1, A_2 \mid r_1)dA_2dA_1 = A_0\int_{K_1}^{\infty}\int_{K_2}^{\infty} \varphi^+(A_1, A_2 \mid r_1)dA_2dA_1$$

where the density is adjusted by the volatility.

Following the same procedure, although tedious, we can derive the three-year zero as:

(C12)
$$\begin{aligned} D(0,T_3) &= P(0,T_1)X_1[\Pi_1^-(K_{13})-\Pi_1^-(K_{12})]+P(0,T_2)X_2[\Pi_2^-(K_{13},K_{23})-\Pi_2^-(K_{12},K_{22})] \\ &+P(0,T_3)X_3\Pi_3^-(K_{13},K_{23},K_{33})+A(0)[\Pi_2^+(K_{12},K_{22})-\Pi_3^+(K_{13},K_{23},K_{33})] \end{aligned}$$

Now, we should observe a pattern for the zero coupon bond prices, which gives (20).

D. Implementation of (20)

The closed form Geske-Johnson model (with constant volatility and constant interest rates) can be computed efficiently only when $n \le 2$. When $n > 2$, then the multi-variate normal probability functions can not be implemented efficiently, particularly in high dimensions. We use the standard equity binomial model with various payoffs to pick up the survival probabilities, zero bond values, and the equity (compound option) value.[23] Note that in the binomial model, there is no need to solve for the implied strike price, K_{ij}. The (compound) option value is directly computed off the actual strikes, X_i.

In (20), we have both volatility and interest rate to be non-constant. We allow volatility to be a deterministic function of time (for calibration) and interest rates to be random (to capture interest rate risk). We shall sketch briefly in this appendix how (20) is implemented.

First, the deterministic volatility function is handled by changing time interval, suggested by Amin (1991). In the binomial model, the up and down sizes are determined by:

(D1)
$$\begin{aligned} u &= e^{\sigma\sqrt{\Delta t}} \\ d &= e^{-\sigma\sqrt{\Delta t}} \end{aligned}$$

If the volatility is changing over time, we simply adjust Δt so that $\sigma\sqrt{\Delta t}$ is constant. In this case, both u and d are constants over time and the tree recombines.

To incorporate stochastic interest rates, we first build a lattice for a "risk-neutral" bivariate Brownian motion process:

(D2)
$$\begin{bmatrix} d\ln A \\ dr \end{bmatrix} = \begin{bmatrix} r-\sigma^2/2 \\ \alpha(\mu-r) \end{bmatrix} dt + \begin{bmatrix} \sigma & 0 \\ 0 & \delta \end{bmatrix} \times \begin{bmatrix} \sqrt{1-\rho^2} & \rho \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} dW_1 \\ dW_2 \end{bmatrix}$$

where $E[dW_1 dW_2] = 0$. We then set up a binomial lattice as follows:

[23] We learned via private conversation that the implementation of the Geske-Johnson model in Eom, Hedweg, and Huang (2002) is computed using the binomial model.

$$\sqrt{\Delta t}$$

$$0$$

$$-\sqrt{\Delta t}$$

We label asset value and interest rate at various nodes as follows:

$$
\begin{array}{ccc}
 & & <\ln A_{22}, r_{22}> \\
 & <\ln A_{11}, r_{11}> & \\
<\ln A_0, r_0> & & <\ln A_{21}, r_{21}> \\
 & <\ln A_{10}, r_{10}> & \\
 & & <\ln A_{20}, r_{20}>
\end{array}
$$

Then we first show that the asset values recombine. Given that:

(D3)
$$
\begin{aligned}
\ln A_{11} &= \ln A_0 + (r_0 - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t} \\
\ln A_{10} &= \ln A_0 + (r_0 - \sigma^2/2)\Delta t - \sigma\sqrt{\Delta t}
\end{aligned}
$$

It is straightforward that:

(D4)
$$
\begin{aligned}
\ln A_{21} &= \ln A_{10} + (r_0 - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t} \\
&= \ln A_{11} + (r_0 - \sigma^2/2)\Delta t - \sigma\sqrt{\Delta t}
\end{aligned}
$$

We now show that the interest rate lattice recombines approximately. Note that:

(D5)
$$
\begin{aligned}
r_{11} &= r_0(1-\alpha\Delta t) + \alpha\mu\Delta t + \delta\sqrt{\Delta t} \\
r_{10} &= r_0(1-\alpha\Delta t) + \alpha\mu\Delta t - \delta\sqrt{\Delta t}
\end{aligned}
$$

Hence,

(D6)
$$
\begin{aligned}
r_{21} &= r_{10}(1-\alpha\Delta t) + \alpha\mu\Delta t + \delta\sqrt{\Delta t} \\
&= [r_0(1-\alpha\Delta t) + \alpha\mu\Delta t - \delta\sqrt{\Delta t}](1-\alpha\Delta t) + \alpha\mu\Delta t + \delta\sqrt{\Delta t} \\
&= r_0(1-2\alpha\Delta t) + 2\alpha\mu\Delta t \\
&= r_{11}(1-\alpha\Delta t) + \alpha\mu\Delta t - \delta\sqrt{\Delta t}
\end{aligned}
$$

This result is an approximated one because higher order terms are ignored.[24]

E. Closed Form Solution for $v(0,T_i)$ *– Rabinovitch* (1989)

Following the interest rate model defined in Appendix D (i.e. the Vasicek model), the zero coupon bond price satisfies the following closed form equation:

(E1) $$P(0,t)=e^{-rF(0,t)-G(0,t)}$$

where

$$F(0,t)=\frac{1-e^{-\alpha t}}{\alpha}$$

$$G(0,t)=(t-F(0,t))\left(\mu-\frac{\delta^2}{2\alpha}\right)+\frac{\delta^2 F(0,t)^2}{4\alpha}$$

Hence, by Ito's lemma we have,

(E2) $$\sigma_P^2(t,T_i)=\left\lfloor\frac{\partial P(t,T_i)}{\partial r}\delta\right|^2=F(t,T_i)^2\delta^2$$

and

(E3) $$\int_0^t \sigma_P^2(u,T_i)du=\int_0^t F(u,T_i)^2\delta^2 du=\delta^2 t+\left((t-2F(0,t))+\frac{1-e^{-2\alpha t}}{2\alpha}\right)\left(\frac{\delta}{\alpha}\right)^2$$

[24] If higher order terms are not ignored, then the lattice does not combine. In the case where the lattice does not recombine, we take the average of two non-recombining values at each node and proceed. We discover that this method produces extremely close results when we compare the zero coupon bond price against the Vasicek closed form model. We use the same technique with both deterministic volatility function and random interest rate are used. Note that in here, since Δt is different period by period, the interest rate dimension of the lattice does not recombine.

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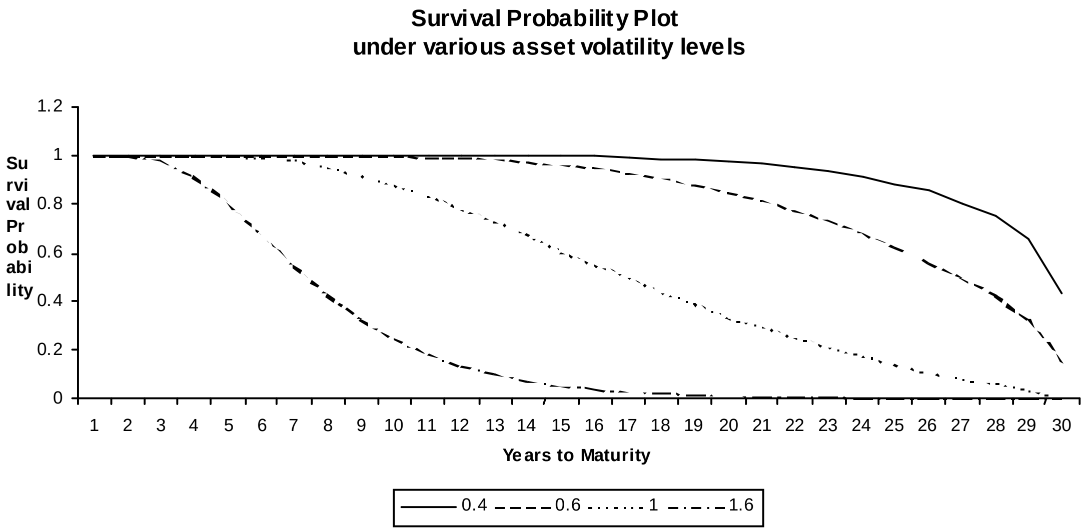
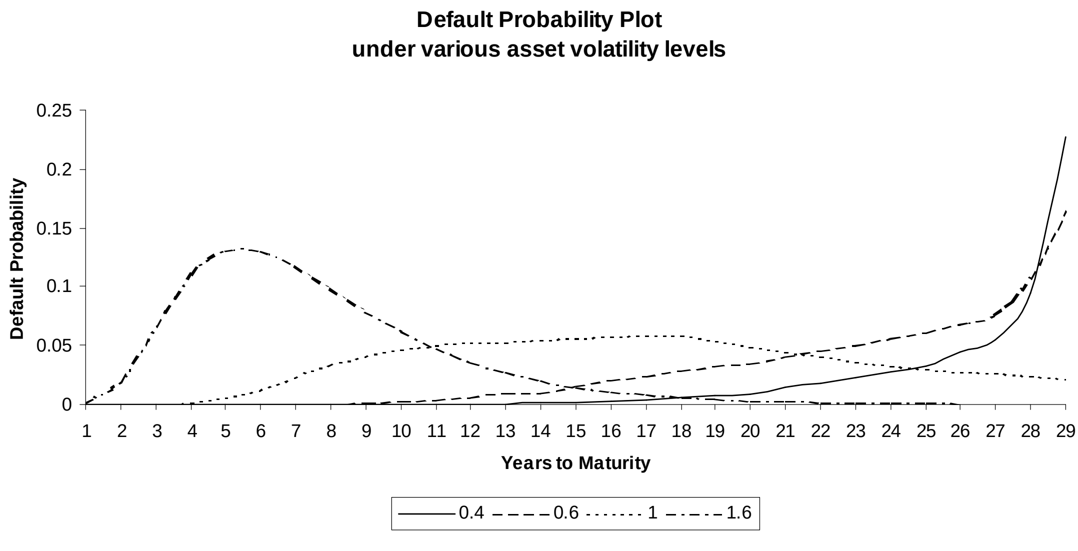
Figure 1: The GJ Model: Zero coupon bond



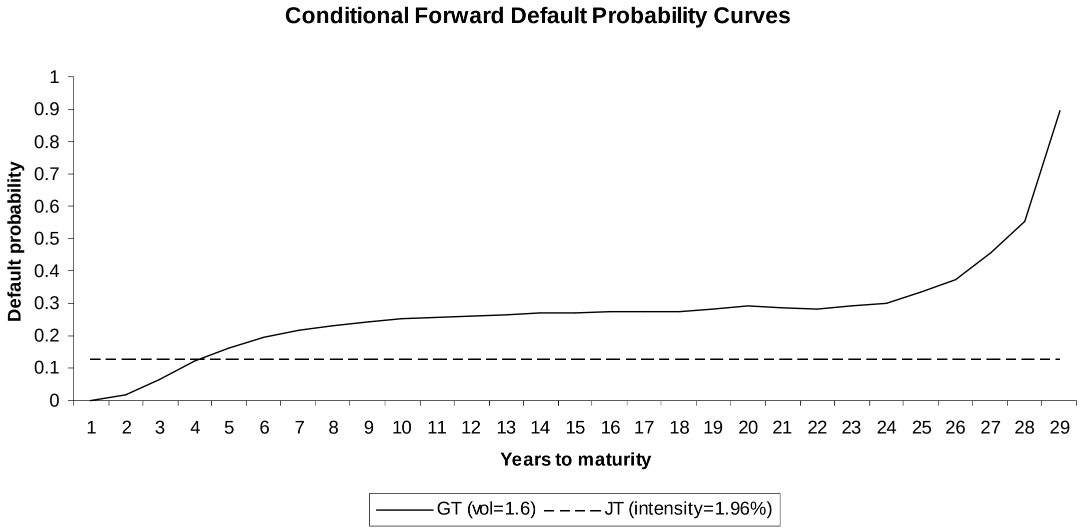
Note: Figure 1 illustrates the survival probability curves under various asset volatility scenarios of the 30-year zero coupon bond under the Geske-Johnson model. The Asset value is set to be $184. The bond has no coupon and a face value of $110. The yield curve is flat at 5%

Figure 1a: The GJ Model: Zero Coupon Bond



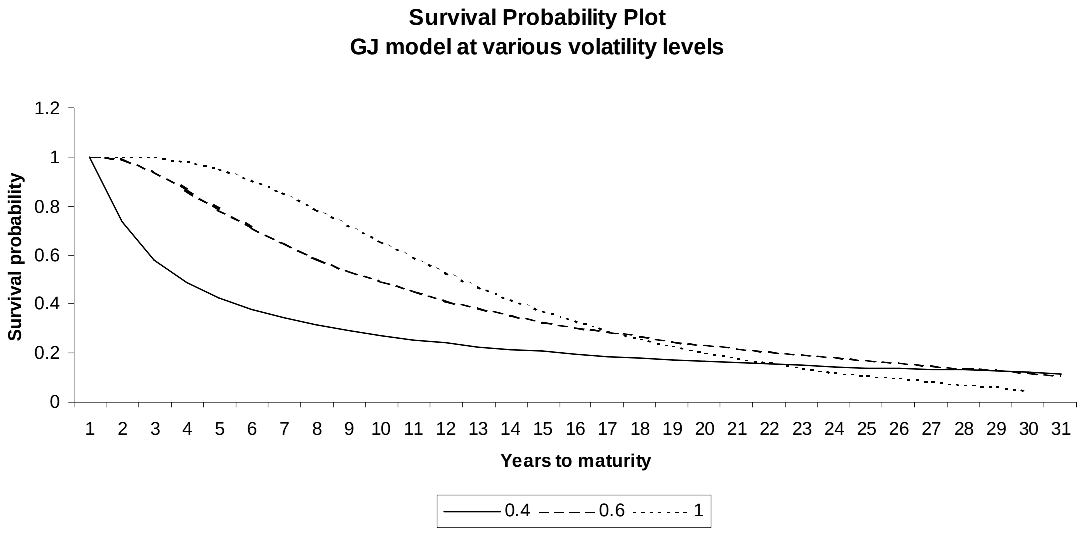
Note: Figure 1a illustrates the default probability curve. All parameters are identical to those in Figure 1.

Figure 1b: Comparison of GJ and JT Models: Zero Coupon Bond



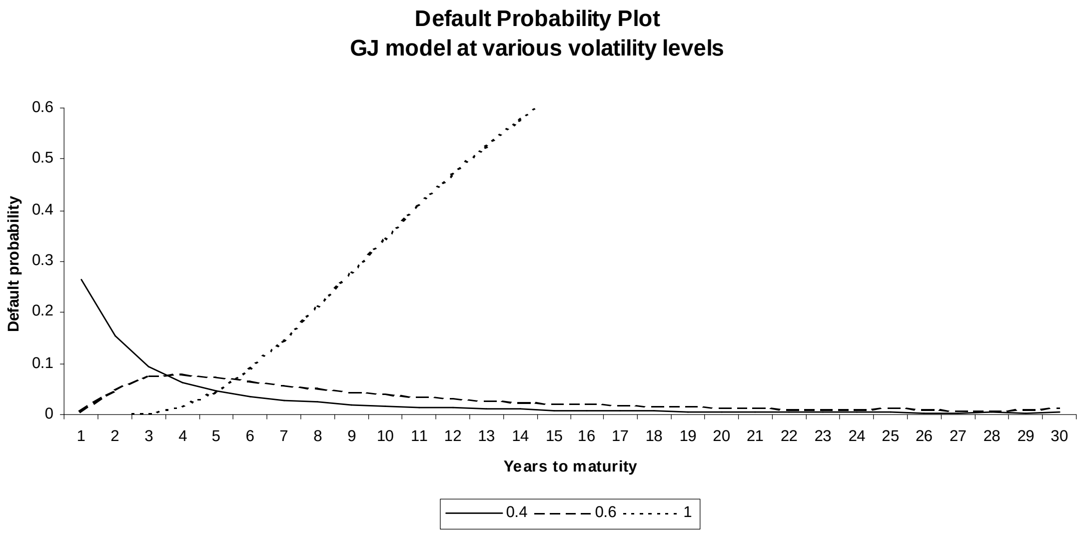
Note: Figure 1b illustrates the default probability curves under GJ and JT models for the volatility level of 1.6. All parameters are identical to those in Figure 1.

Figure 2: The GJ model: 10% Coupon Bond



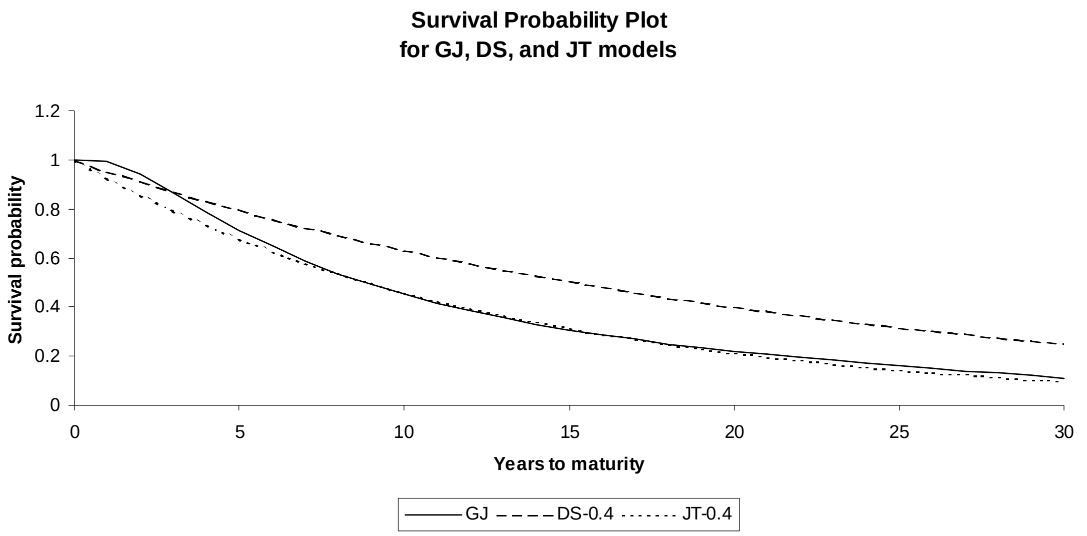
Note: Figure 2 illustrates the survival probability curves under various asset volatility scenarios of the 30-year Geske-Johnson model. The Asset values are 123, 290, 15,000 for volatility levels of 0.4, 0.6, and 1.0 repsectively. The bond has a coupon of $10 and price of par.

Figure 2a: The GJ Model: 10% Coupon Bond



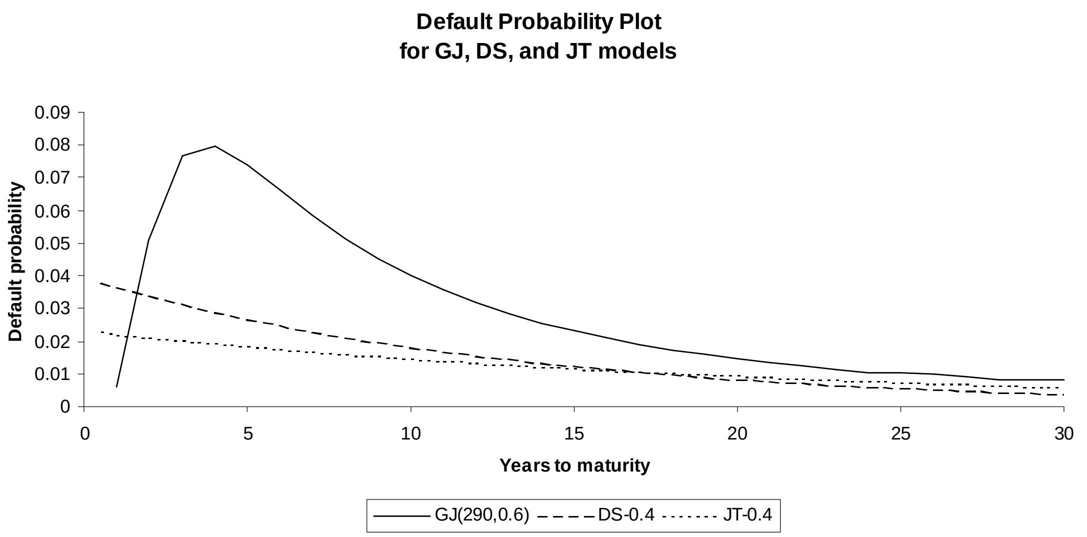
Note: Figure 2a illustrates the default probability curves under various asset volatility scenarios of the 30-year Geske-Johnson model. The Asset values are 123, 290, 15,000 for volatility levels of 0.4, 0.6, and 1.0 repsectively. The bond has a coupon of $10 and price of par. All parameters are identical to those in Figure 2.

Figure 3: Comparison of the GJ, DS and JT Models: Par Coupon Bond



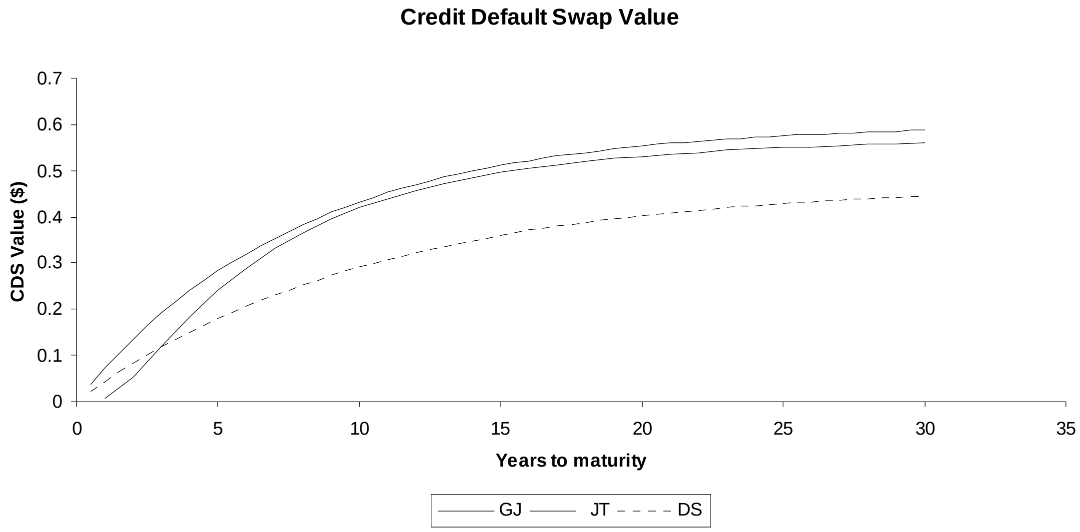
Note: Figure 3 illustrates the survival probability curves under the 30-year Geske-Johnson, Duffie-Singleton with 0.4 recovery ratio, and Jarrow-Turnbull with 0.4 recovery ratio models. The Asset value is set to be \$290 and volatility 0.6 so that the bond is priced at par. The bond has \$10 coupon and a face value of \$110.

Figure 3a: (Comparison of the GJ, DS and JT Models: Par Coupon Bond



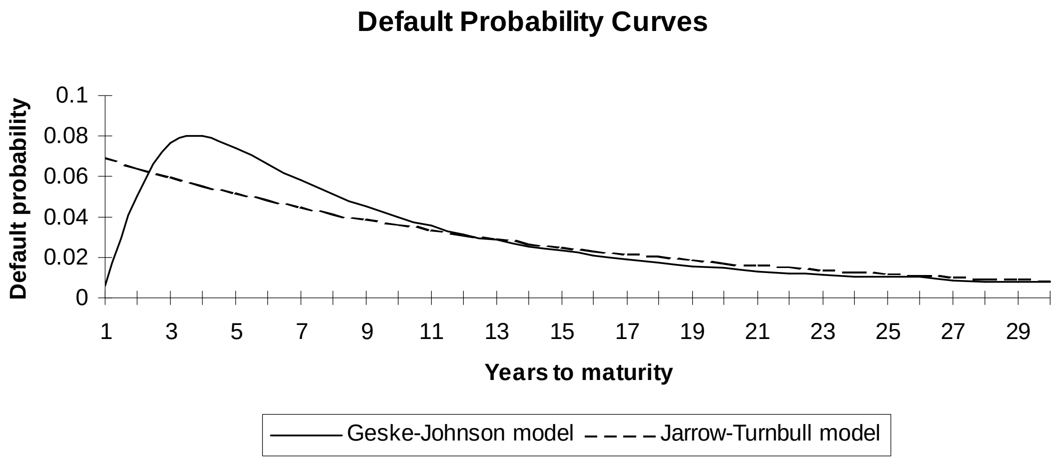
Note: Figure 3a illustrates that under a 10% coupon bond, the Geske-Johnson model can generate desired default probability curve. All parameters are identical to those in Figure 3.

Figure 4: Credit Default Swap Values for the GJ, JT, and DS models



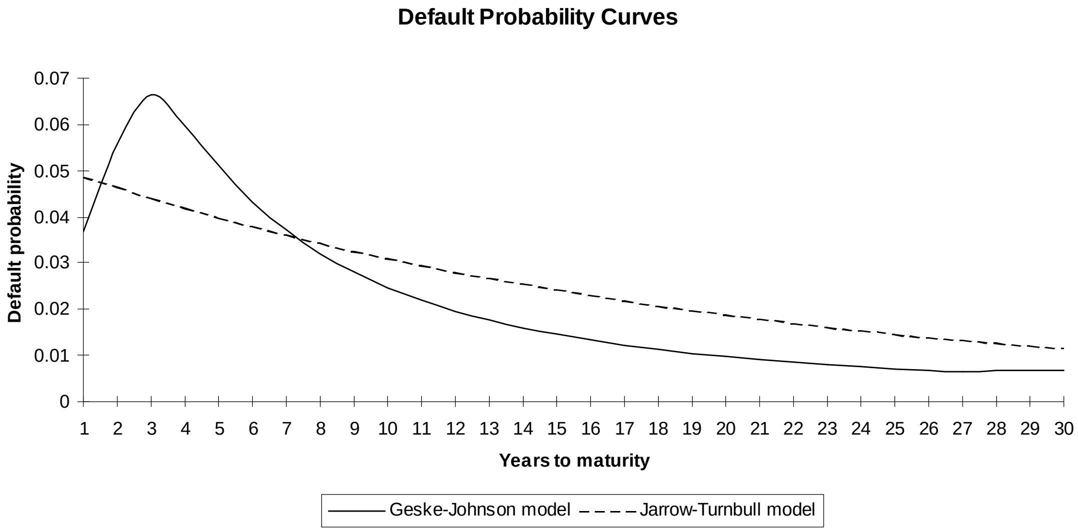
Note: Data that compute this figure are taken from those generate Figure 3a.

Figure 5: (Unconditional) Default Probability Curve for the GJ and JT models



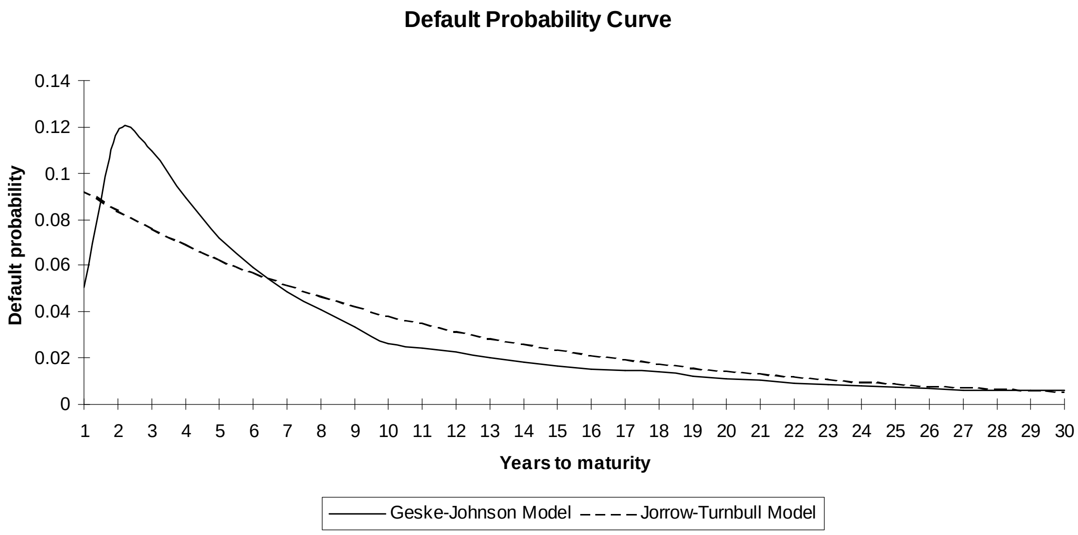
Note: In the Geske-Johnson model, asset=290, volatility=0.6 (so that debt=100), coupon=10, face=100, and yield curve is flat at 5%. The credit default swap value is $32.87. A recovery rate that satisfies a flat recovery is 0.5840. The intensity value is 7.19% and the bond price is 110.1.

Figure 6: (Unconditional) Default Probability Curve for the GJ and JT models



Note: In the Geske-Johnson model, asset=184, volatility=0.4 (so that debt=100), coupon=10, face=100, and yield curve is generated by the Vasicek model given in the text. The credit default swap value is $11.18. A recovery rate that satisfies a flat recovery is 0.3117. The intensity value is 4.99% and the bond price is 83.74.

Figure 7: (Unconditional) Default Probability Curve for the GJ and JT models



Note: In the Geske-Johnson model, asset=184, volatility=0.6 (debt=80), coupon=10, face=100, and yield curve is generated by the Vasicek model given in the text. The credit default swap value is $29.06. A recovery rate that satisfies a flat recovery is 0.5485. The intensity value is 9.84% and the bond price is 81.05.