Modern Portfolio Theory
with
Homogeneous Risk Measures

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Abstract

The Modern Portfolio Theory for investments by Harry Markowitz is usually associated with mean-variance return-risk analysis. Nevertheless, even Markowitz himself suggested the use of semi-variance instead of variance since the variance has the obvious weakness to penalize potentials for high gains to the same degree as potentials for high losses. Variance and standard deviation were preferred as risk measures mainly for their computational advantages. This reason became more and more obsolete over the years through the ever growing computer power now available to everyone. Management attention turned to other risk measures, in particular to VaR (Value-at-Risk) and its relatives.

This development motivates a revision of Modern Portfolio Theory. We examine which consequences of assuming variance or standard deviation as risk measure remain true when switching to other measures. It turns out that two-funds separation is linked to homogeneity of the risk measure. Therefore CAPMs (Capital Asset Pricing Models) can be derived for differentiable, homogeneous risk measures. Concavity of the risk-return efficient frontier requires convexity of the underlying risk measure. Moreover, in case of an homogeneous and convex risk measure, there is a unique CAPM market portfolio.

These observations point out once more the importance of homogeneity and convexity properties for risk measures. In this sense, CVaR (Conditional Value-at-Risk) is an optimal representative for the family of VaR-related risk measures.

Contents

1. Markowitz model of investment

Setting

Portfolio with 1 risk-less asset, d risky assets

Random excess returns over risk-free rate:
$X = (X_1, \ldots, X_d)$ (0 for risk-less asset)

Expected returns: $m = (m_1, \ldots, m_d)$, all $m_i > 0$

Asset weights: h for risk-less asset
$u = (u_1, \ldots, u_d)$ for risky assets

Excess return to portfolio: $X'u = \sum_{i=1}^d X_i u_i$

Expected return to portfolio: $m'u = \sum_{i=1}^d m_i u_i$

Risk measure $\rho(h,u) \geq 0$ with $\rho(0,0) = 0$,
positively homogeneous of order $a > 0$,
i.e. $\rho(th, tu) = t^a \rho(h,u)$ for $t > 0$

Examples of risk measures

Standard deviation: $\rho_{\text{st}}(h,u) = \sqrt{\text{var}[X'u]}$

Lower semi-standard-deviation: $x^- = \begin{cases} -x, & x < 0 \\ 0\ , & x \geq 0 \end{cases}$

$$\rho_{\text{semi}}(h,u) \ = \ \sqrt{\text{E}[((X'u - m'u)^-)^2]}$$

Lower partial a-moment with target τ:

$$\rho_{\text{part}}(h,u) \ = \ \text{E}\left[\left((X'u - \tau \textstyle\sum_{i=1}^{d} u_i - \tau h)^-\right)^a\right]$$

Value-at-Risk (VaR) at level $\alpha \in (0,1)$:

$$\rho_{\text{VaR}}(h,u) \ = \ m'u - q_\alpha(X'u),$$

with $q_\alpha(\xi) \ = \ \inf\{x \in \mathbb{R} \ : \ \text{P}[\xi \leq x] \geq \alpha\}$

Conditional Value-at-Risk (CVaR) at level $\alpha \in (0,1)$:

$$\rho_{\text{CVaR}}(h,u) \ = \ m'u - \text{E}\left[X'u \,|\, X'u \leq q_\alpha(X'u)\right]$$

2. Portfolio selection

Problems

Total investment $c > 0$.
No short-selling of risky assets, i.e. $u_1, \ldots, u_d \geq 0$.
Fixed level $r > 0$ of risk.

Find allocation u^* such that

(i) $\rho(c - \sum_{i=1}^{d} u_i^*, u^*) = r$

(ii) $m'u^* = \max\left\{ m'u : \rho(c - \sum_{i=1}^{d} u_i, u) = r \right\}$

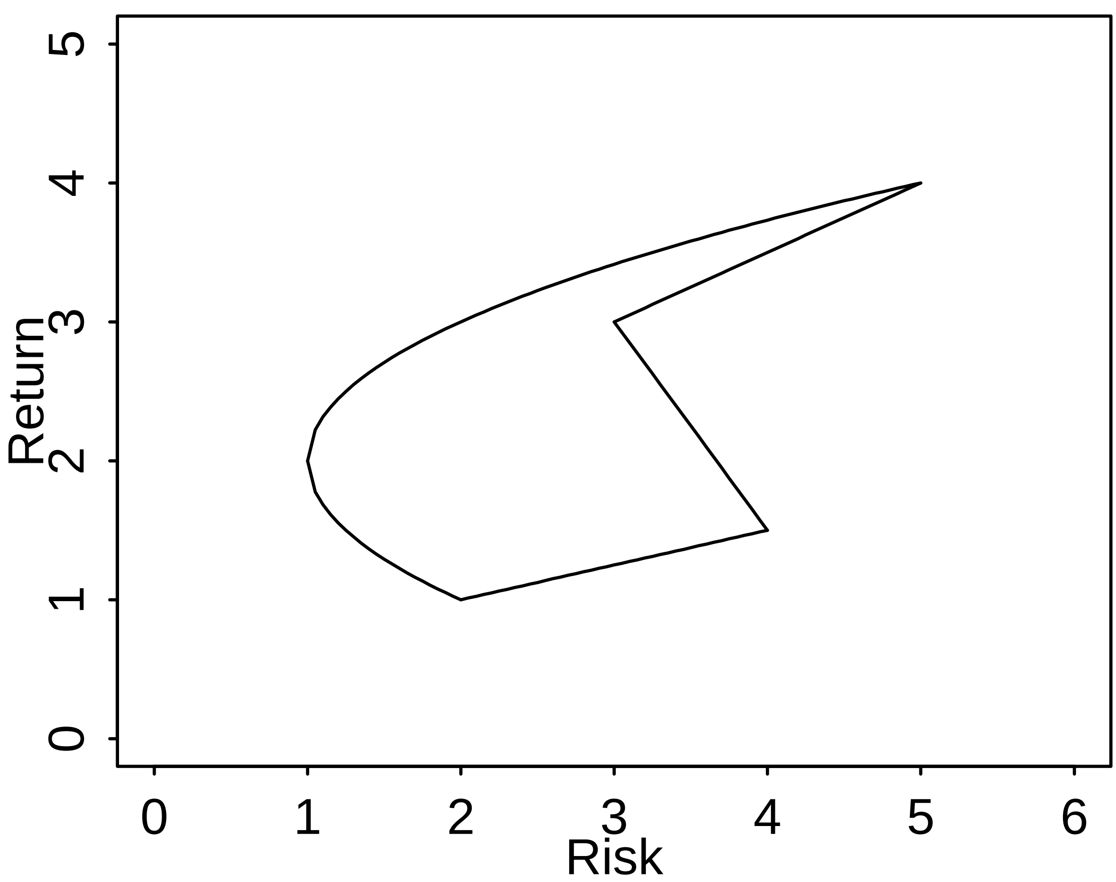
Case 1: with borrowing and lending,
i.e. $c - \sum_{i=1}^{d} u_i < 0$ allowed

Case 2: without borrowing,
i.e. $c - \sum_{i=1}^{d} u_i \geq 0$

Visualization 1

Feasible portfolios for total investment c without borrowing and lending

$$F(c) = \left\{ (\rho(0,u), m'u) \,:\, u \in [0,\infty)^d, \textstyle\sum_{i=1}^d u_i = c \right\}$$

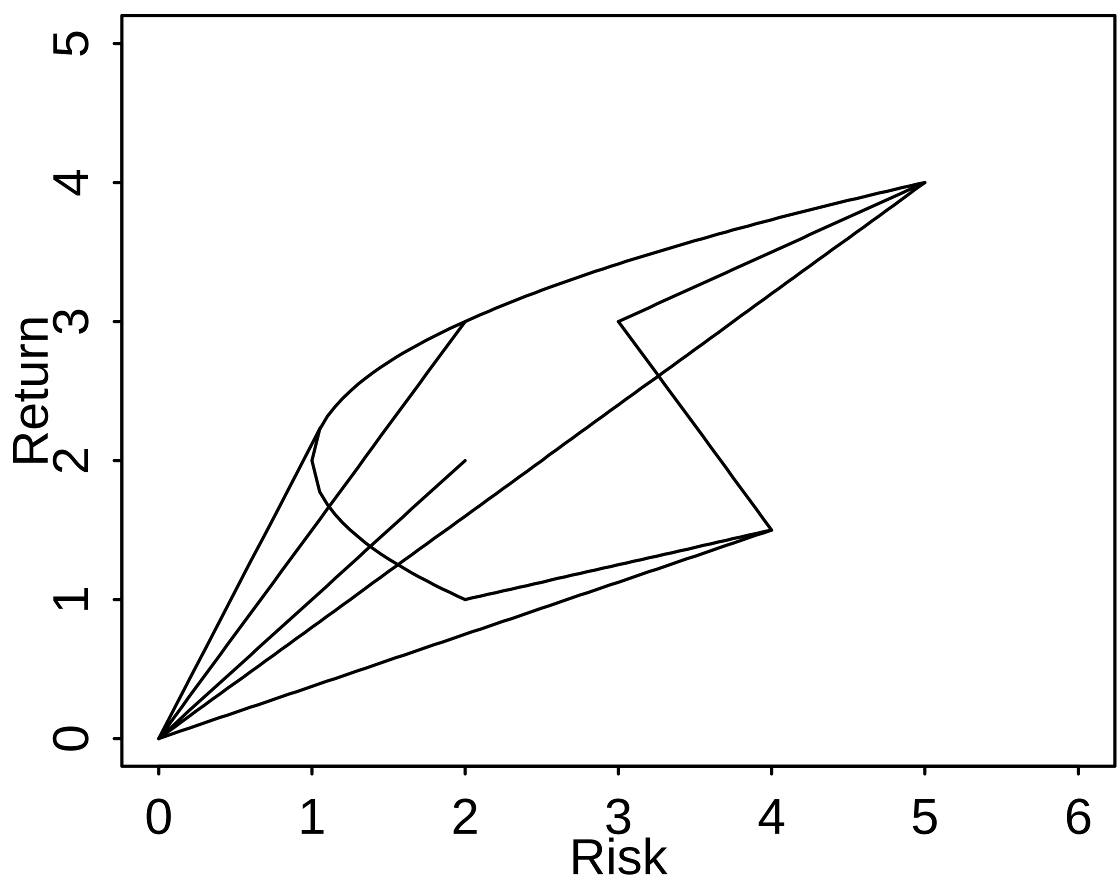


Visualization 2

Feasible portfolios for total investment c with lending

$$F_{\mathrm{l}}(c) = \Big\{ (\rho(c - \textstyle\sum_{i=1}^d u_i, u), m'u) : \\ u \in [0,\infty)^d,\ \textstyle\sum_{i=1}^d u_i \le c \Big\}$$

Assumption: $\rho(h,u) = \rho(0,u)$,
homogeneous of order 1

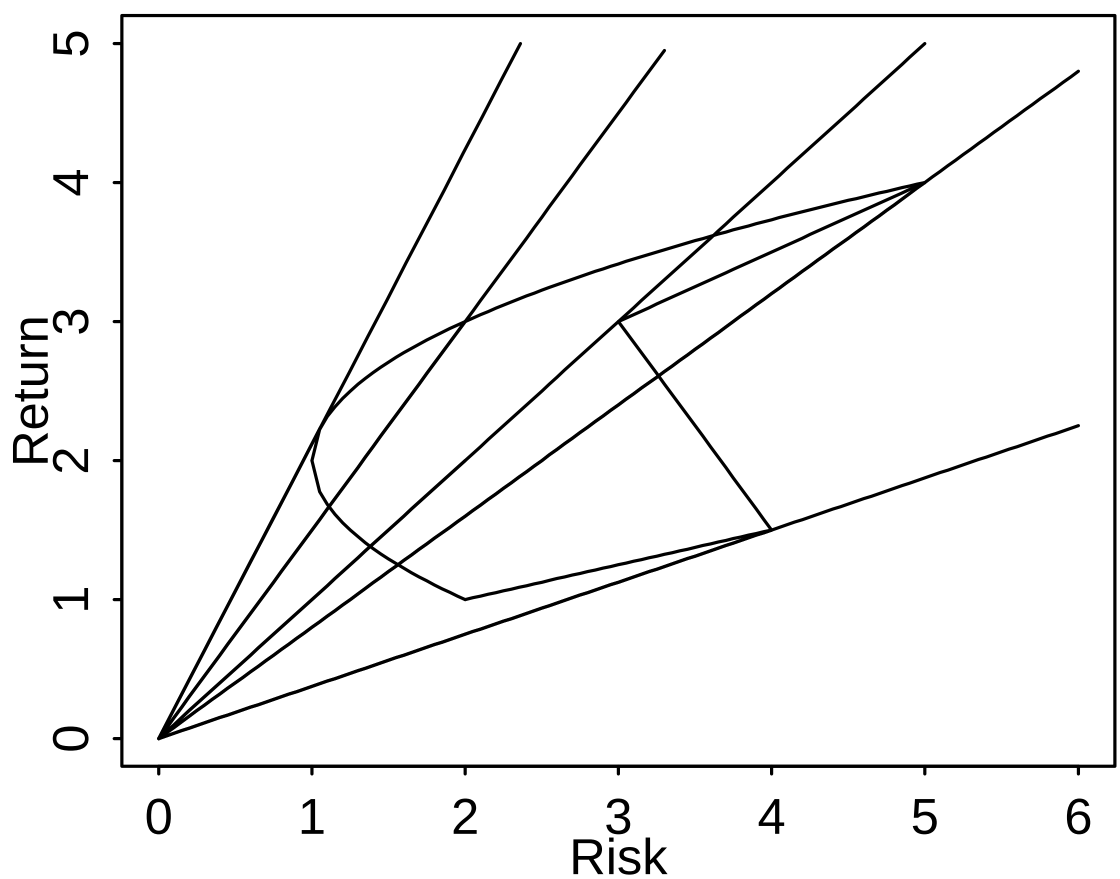


Visualization 3

Feasible portfolios for total investment c with lending and borrowing

$$F_{\text{lb}}(c) \quad = \quad \left\{(\rho(c - \textstyle\sum_{i=1}^{d} u_i, u), m'u) \; : \; u \in [0, \infty)^d\right\}$$

Assumption: $\rho(h, u) = \rho(0, u)$,
homogeneous of order 1



3. Two-funds separation and CAPM

Portfolio selection with borrowing/lending

Assumption: $\rho(h,u) = \rho(0,u)$,
positively homogeneous of order $a > 0$

Proposition Let $u^* \in [0,\infty)^d$ with $\rho(0,u^*) > 0$.
Then

$$\frac{(m'u^*)^a}{\rho(0,u^*)} \;=\; \max\left\{\frac{(m'u)^a}{\rho(0,u)} \;:\; u \text{ with } \rho(0,u) > 0\right\}$$

if and only if
for all $c > 0$ and $r > 0$ there is some $t > 0$ such that

(i) $\rho(c - t\sum_{i=1}^d u_i^*, t\,u^*) \;=\; r$

(ii) $t\,m'u^* \;=\; \max\left\{m'u \;:\; \rho(c - \sum_{i=1}^d u_i, u) = r\right\}$.

Remark $S_\rho(u) = \frac{(m'u)^a}{\rho(0,u)}$ is a *Generalized Sharpe Ratio*.

Capital Asset Pricing Model (CAPM)

Lemma Let $u \mapsto \rho(0,u)$ be continuous and positive on $\Delta_d = \{u \in [0,\infty)^d \ : \ \sum_{i=1}^d u_i = 1\}$. Then there is some $u^* \in \Delta_d$ which maximizes $S_\rho(u)$ on $[0,\infty)^d \backslash \{0\}$.

u^* is called *two-funds separating.*

Proposition Let $u \mapsto \rho(0,u)$ be partially differentiable in $u^* \in (0,\infty)^d$ and u^* with $\rho(0,u^*) > 0$ be maximizing for $S_\rho(u)$. Then for $i = 1,\dots,d$

$$m_i \quad = \quad m'u^* \frac{\frac{\partial \rho}{\partial u_i}(0,u^*)}{a\,\rho(0,u^*)}.$$

Remarks

- Under the usual CAPM assumptions, u^* is the allocation of the *Market Portfolio*.
- $\frac{\frac{\partial \rho}{\partial u_i}(0,u^*)}{a\,\rho(0,u^*)}$ is a *Generalized* β of asset i.

Derivatives of risk measures

$$
\begin{aligned}
\frac{\partial \rho_{\mathrm{st}}}{\partial u_i}(h,u) &= \frac{\mathrm{cov}[X_i, X'u]}{\sqrt{\mathrm{var}[X'u]}} \\
\frac{\partial \rho_{\mathrm{semi}}}{\partial u_i}(h,u) &= -\frac{\mathrm{cov}[X_i, (X'u - m'u)^-]}{\sqrt{\mathrm{E}[((X'u - m'u)^-)^2]}} \\
\frac{\partial \rho_{\mathrm{VaR}}}{\partial u_i}(h,u) &= m_i - \mathrm{E}\left[X_i \,|\, X'u = q_\alpha(X'u)\right] \\
\frac{\partial \rho_{\mathrm{CVaR}}}{\partial u_i}(h,u) &= m_i - \mathrm{E}\left[X_i \,|\, X'u \leq q_\alpha(X'u)\right]
\end{aligned}
$$

Assumption for ρ_{VaR} and ρ_{CVaR}:
X_1 has a continuous conditional density given $(X_2, \ldots, X_d)$.

3. Two-funds separation and CAPM

Derivatives of lower partial moments

Case $a > 1$:

$$\frac{\partial \rho_{\text{part}}}{\partial u_i}(h,u) = a\,\mathrm{E}\Big[(\tau - X_i) \times \Big(\big(X'u - \tau \textstyle\sum_{i=1}^d u_i - \tau h\big)^-\Big)^{a-1}\Big]$$

$$\frac{\partial \rho_{\text{part}}}{\partial h}(h,u) = a\tau\,\mathrm{E}\Big[\Big(\big(X'u - \tau \textstyle\sum_{i=1}^d u_i - \tau h\big)^-\Big)^{a-1}\Big]$$

Case $a = 1$:

$$\frac{\partial \rho_{\text{part}}}{\partial u_i}(h,u) = \mathrm{E}\Big[(\tau - X_i)\,\mathbb{I}_{\{X'u \le \tau \sum_{i=1}^d u_i + \tau h\}}\Big]$$

$$\frac{\partial \rho_{\text{part}}}{\partial h}(h,u) = \tau\,\mathrm{P}\Big[X'u \le \tau \textstyle\sum_{i=1}^d u_i + \tau h\Big]$$

Assumption for case $a = 1$:

X_1 has a continuous conditional density given $(X_2, \ldots, X_d)$.

Portfolio selection without borrowing

Proposition Let $u \mapsto \rho(0,u)$, $u \in \Delta_d(c) = \{v \in [0,\infty)^d : \sum_{i=1}^d v_i = c\}$ be convex and continuous. Let

$$\begin{aligned} m^* &= \max\{m'u : u \in \Delta_d(c)\} \\ r_* &= \min\{\rho(0,u) : u \in \Delta_d(c)\} \\ r^* &= \min\{\rho(0,u) : u \in \Delta_d(c),\ m'u = m^*\}. \end{aligned}$$

Define $M : [r_*, r^*] \to \mathbb{R}$ by

$$M(r) = \max\{m'u : u \in \Delta_d(c),\ \rho(0,u) = r\}.$$

If $r_* < r^*$ then M is non-decreasing and concave.

Remarks

- $\{(r, M(r)) : r \in [r_*, r^*]\}$ is a concave *Efficient Frontier*.
- If $\rho(h,u)$ is positively homogeneous of order 1 then $\rho(0,u)$ is convex in u if and only if $\rho(0,u)$ is sub-additive, i.e. $\rho(0,u+v) \le \rho(0,u) + \rho(0,v)$.
- ρ_{st} and ρ_{CVaR} are sub-additive.

Characterization of sub-additivity

Lemma Let $u \mapsto \rho(0,u)$ be positively homogeneous of order 1 and partially differentiable. Then $\rho(0,u)$ is sub-additive if and only if for all u, v

$$u'\nabla\rho(0,u+v) \quad \leq \quad \rho(0,u)\,. \tag{1}$$

Remarks

- By homogeneity:

$$\rho(0,u+v) = u'\nabla\rho(0,u+v) + v'\nabla\rho(0,u+v)\,.$$

  Interpretation: $u'\nabla\rho(0,u+v)$ is contribution of sub-portfolio with allocation u to total risk $\rho(0,u+v)$ of portfolio.
- Interpretation of (1): Risk contribution of sub-portfolio is always smaller than its stand-alone risk.
- ρ positively homogeneous of order 1 and sub-additive $\Rightarrow$ Generalized Sharpe Ratio $S_\rho(u)$ has at most one maximum.

5. Conclusion

Important features of the Modern Portfolio Theory are valid for general homogeneous risk measures.
CVaR is homogeneous of order 1 and sub-additive as it is the standard deviation. Interpretation of CVaR is more intuitive than that of stand deviation.

Caveat: Sub-additivity of CVaR breaks down when the distribution of the portfolio return is discontinuous. Differentiability of CVaR with respect to the asset weights requires some conditions (not too restrictive) on the joint distribution of $(X_1, \ldots, X_d)$.

References

D. Bertsimas, G.J. Lauprete, A. Samarov (2000) Shortfall as a risk measure: properties, optimization and applications. Working paper, Sloan School of Management, MIT.

D. Tasche (1999) Risk contributions and performance measurement. Working paper, TU München.

D. Tasche (2000) Conditional expectation as quantile derivative. Working paper, TU München.