Notes on Intertemporal Optimization
Econ 204A - Henning Bohn

Most of modern macroeconomics involves models of agents that optimize over time. The basic ideas and tools are the same as in microeconomics, notably the Lagrange multiplier technique. At a technical level, the main differences are interpretational or notational: In macroeconomics, goods are usually “dated commodities” (see Debreu, Theory of Value, ch.2) that are consumed at different times and their relative prices can often be expressed in terms of interest rates or discount factors. Part 1 of this note will explain the linkage. In terms of economic intuition, time adds two interesting dimensions. First, it is sometimes instructive to keep track of agents’ asset positions as time passes. Second, with uncertainty, new information may arrive that enables agents to make purchases of later-dated commodities contingent on information on which early consumption cannot be conditioned. Part 2 of this note will explain the relation between overall (“intertemporal”) budget constraints and the period-by-period budget equations designed to track assets positions. To avoid “math overload,” stochastic issues will be deferred until later. Part 3 will survey alternative solution techniques and Part 4 examines infinite horizon problems.

Micro and Macro Problems:
Our first objective is to see that the intertemporal decision problems in macroeconomics are in principle nothing new. With a little re-labeling, they fit nicely into the standard Lagrangian optimization approach familiar from microeconomics courses.

Consider the following microeconomics problem. A consumer has preferences over N commodities, \( u(u(c_1,\ldots,c_N)) \), increasing and concave, where \( (c_1,\ldots,c_N) \) is the consumption bundle. The consumer has endowments \( (y_1,\ldots,y_N) \) of these commodities. On an individual level, goods prices \( (p_1,\ldots,p_N) \) are given. Then the utility-maximization problem is
Problem#1:

Maximize \( u = u(c_1, \ldots, c_N) \)

over choice variables \( c_1, \ldots, c_N \),

\[
(1) \quad \sum_{i=1}^{N} p_i \cdot c_i = \sum_{i=1}^{N} p_i \cdot y_i.
\]

The summation notation will be used frequently: For any variable \( x \), read \( \sum_{i=1}^{N} x_i = x_1 + \ldots + x_N \).

With \( \lambda \) as the Lagrange multiplier on the constraint, the first order conditions (FOC for short) are

\[
\frac{\partial u}{\partial c_i} = \lambda \cdot p_i, \quad i=1, \ldots, N
\]

where subscripts denote partial derivatives (as in \( u_i = \frac{\partial u}{\partial c_i} \)). Provided \( p_1 \neq 0 \), this can be simplified to

\[
(2) \quad \frac{u_i}{u_1} = \frac{p_i}{p_1} \quad \text{for } i=2, \ldots, N.
\]

That is, consumption depends only on relative prices. The constraint (1) and the FOC (2) form a system of \( N \) equations in the \( N \) choice variables \( c_1, \ldots, c_N \). Often, the system cannot be solved in closed form. We can, however, say something about the structure of the solution.

The resulting choice vector \( (c_1, \ldots, c_N) \) is a function of the exogenous variables of the problem, endowments and relative prices. Note that endowments only enter through the r.h.s. of (1). It is therefore convenient to define “wealth” \( W = \sum_{i=1}^{N} p_i \cdot y_i \). Then the problem yields a system of demand functions:

\[
c_1 = c_1(W/p_1, p_2/p_1, \ldots, p_N/p_1)
c_2 = c_2(W/p_1, p_2/p_1, \ldots, p_N/p_1)
\ldots
\]

\[
c_N = c_N(W/p_1, p_2/p_1, \ldots, p_N/p_1).
\]

Often, utility is only well-defined for non-negative consumption. To be precise one should then add the non-negativity conditions \( c_i \geq 0, \ i=1, \ldots, n, \) to Problem#1 and work with the relevant Kuhn-Tucker conditions. In macro, we will usually skip these complications and implicitly assumed that \( u(\cdot) \) is such that we can focus on interior solutions.

For the special case of \( N=2 \), the consumer problem reduces to

\[
\text{MAX: } u(c_1, c_2) \quad \text{s.t. } p_1 \cdot c_1 + p_2 \cdot c_2 = W
\]
and can be graphed in the usual \((c_1, c_2)\) indifference curve diagram.

So much for microeconomics. Please alert me or the TA if the above problem is not an easy problem for you. To test your abilities, derive the demand functions for the case of log-utility.

Turning to macro, consider an economy with a large number of agents. To start, assume the economy operates for \(T+1\) periods only, periods \(t=0, \ldots, T\). Agents have preferences over consumption \(c_t\) in each period, \(u=u(c_0, \ldots, c_T)\), and they have incomes \(y_t\), \(t=0, \ldots, T\). (Think of per-period consumption as a fixed-weight basket of different physical commodities; one could keep track of different commodities, but that would distract from the macro issues.) Let \(p_t\) be the period-0 price of a discount bond with a payoff in period \(t\) equal to the cost of one unit of consumption \(c_t\); define \(p_0=1\). Then \(p_t/p_0\) is the relative price of period-\(t\) over period-0 consumption and the consumer budget constraint can be written as

\[
\sum_{t=0}^{T} p_t \cdot c_t = \sum_{t=0}^{T} p_t \cdot y_t.
\]

That is, the present value of consumption equals the present value of income. This constraint is known as the **intertemporal budget constraint** (IBC). The consumer maximization problem is then

**Problem#2:**

Maximize \(u=u(c_0, \ldots, c_T)\)

over choice variables \(c_0, \ldots, c_T\),

\[
(3) \quad \text{s.t. } \sum_{t=0}^{T} p_t \cdot c_t = \sum_{t=0}^{T} p_t \cdot y_t
\]

Note that Problem#1 and Problem#2 are virtually identical, except for slightly different notation. The lesson is that you should feel comfortable applying all your microeconomics knowledge to macroeconomics. For example, for given specifications of the utility function, you should be able to derive demand functions expressing period-\(t\) consumption as function of the relative prices and of the present value of income.

Sometimes, you will have to do a little algebra to express a macro problem in the above format. Notably, it is often convenient in macro to express relative prices in terms of interest rates and discount factors. Let \(r_t\), \(t=1, \ldots, T\), be a sequence of exogenous market interest rates, where \(r_t\) is the real interest rate on a bond issued in period \(t\) that matures in period
t+1. That is, consumers obtain \((1+r_t)\) units of period-(t+1) consumption for reducing period-\(t\) consumption by 1 unit. Then the relative cost of period-(t+1) consumption \(c_{t+1}\) in terms of period-\(t\) consumption is
\[
\rho_t = \frac{p_t}{p_{t-1}} = \frac{1}{1+r_t},
\]
the period-\(t\) discount factor. Going back from \(t\) to \(t-1\), to \(t-2\), and so on to period-0, one finds that the cost of period-\(t\) consumption \(c_t\) relative to period-0 consumption \(c_0\) is
\[
(4) \quad p_t = \rho_t \cdot p_{t-1} = \rho_t \cdot \rho_{t-1} \cdot \ldots \cdot \rho_1 = \prod_{j=1}^{t} \frac{1}{1+r_j}
\]
The lesson is that all the relative prices \(p_t\) in Problem#2 above can be expressed in terms of interest rates and/or discount factors.

Asset Accumulation:

Thinking about an intertemporal optimization problem as a re-labeled static problem is useful from a technical perspective, but it does not do justice to the dynamic problem. Consumption and other activities happen sequentially; the timing of consumption and income is usually not perfectly synchronized, forcing consumers to borrow or lend on financial markets; and the observable dynamics of asset and debt accumulation are often the economically most interesting aspect of an intertemporal problem. Because of economists' interest in asset positions, intertemporal problems are often set up in a way that looks quite different than Problem#2 above.

The alternative starting point is a series of period-by-period budget equations that trace out the process of asset accumulation. Let \(b_t\) be the bond holdings (positive or negative; if negative, meaning loans) of an agent at the end of period \(t\), \(t=0,\ldots,T\). Given initial assets \((1+r_0) \cdot b_{t-1}\) at the start of period \(t\), income \(y_t\), and consumption \(c_t\), we obtain a end-of-period assets as sum of savings \(y_t - c_t\) and initial assets,
\[
(5) \quad b_t = y_t - c_t + (1+r_t) \cdot b_{t-1}.
\]
If we define \((1+r_0) \cdot b_{-1} = 0\), this equation applies for all periods \(t=0,\ldots,T\). The individual choice variables are then sequences of \(b_t\) and \(c_t\).

Note that equations (5) are mere accounting identities but not really constraints, because they do not impose a limit on the level of \(b_t\). To constrain agents' choices, an end-point restriction must be imposed. The natural one is \(b_T \geq 0\), which prevents agents from borrowing without repayment.
To get rid of the inequality (which would require a Kuhn-Tucker approach), note that rational consumers do not leave positive assets at the end. Hence, one may treat the endpoint restriction as an equality: \( b_T = 0 \).

Then one can formulate the consumer optimization problem as

**Problem 3:**

Maximize \( u = u(c_0, \ldots, c_T) \)

over choice variables \( c_0, \ldots, c_T, b_0, \ldots, b_{T-1} \),

s.t.

\[ b_t = y_t - c_t + (1 + r_t) \cdot b_{t-1} \quad \text{for } t = 0, \ldots, T \]

where \( b_T = 0, b_{-1} = 0 \) are given.

This looks like a more complicated problem than Problem 2 because it involves more variables and equations: Problem 2 has \( T+1 \) choice variables and 1 constraint. Problem 3 has \( 2 \cdot T+1 \) choice variables and \( T+1 \) constraints. On the other hand, Problem 3 provides more insights about what the consumer actually does over time, how savings and bond holdings evolve.

Our objective is to show that Problems 2 and 3 are actually identical.\(^1\) To see the equivalence, multiply (5) by the relative price \( p_t \) and exploit (4),

\[
\begin{align*}
 p_t \cdot b_t &= p_t \cdot y_t - p_t \cdot c_t + p_t \cdot [(1 + r_t) \cdot b_{t-1}] \\
 &= p_t \cdot y_t - p_t \cdot c_t + p_t \cdot \frac{p_{t-1}}{p_t} \cdot b_{t-1} \\
 &= p_t \cdot y_t - p_t \cdot c_t + p_{t-1} \cdot b_{t-1}
\end{align*}
\]

This can be interpreted as the budget equation discounted back to period-0.

Then sum over \( t \),

\[
\sum_{t=0}^{T} [p_t \cdot b_t] = \sum_{t=0}^{T} [p_t \cdot y_t] - \sum_{t=0}^{T} [p_t \cdot c_t] + \sum_{t=0}^{T-1} [p_t \cdot b_t]
\]

keeping in mind that \( b_{-1} = 0 \). Canceling the offsetting part of the \( p_t \cdot b_t \) sums, one finds

\[
(6) \quad p_T \cdot b_T = \sum_{t=0}^{T} [p_t \cdot y_t] - \sum_{t=0}^{T} [p_t \cdot c_t]
\]

So far, we have only worked with accounting identities, equations (4) and (5). Equation (6) shows that the budget equations (5) are equivalent to the

\(^1\) If one treated \( b_T \) as choice variable, too, one more choice variable and one more constraint, \( b_T \geq 0 \), would have to be added; that would be equivalent, too. I leave it as an exercise to show that the optimal solution will always satisfy \( b_T = 0 \), provided utility is strictly increasing.
inter-temporal budget constraint (3) if and only if \( p_T \cdot b_T = 0 \). Hence, the endpoint restriction \( b_T = 0 \) is crucial (assuming \( p_T \neq 0 \)). With this conditions, Problems #2 and #3 are equivalent because they have the equivalent constraints.

Now suppose you are given an optimization problem like #3. Doing the steps from eq.(5) to eq.(6) in reverse, you can always substitute away the asset positions and solve the problem as a pure consumption problem of the form #2, using your knowledge of static optimization. Given the optimal consumer demand functions, you can read off the asset positions from eq.(5).

With regard to Problem #3, you may wonder why anyone would ever try to solve this problem rather than transform it into Problem #2. One answer is that the FOC of Problem #3 involves Lagrange multipliers that have a useful interpretation as shadow values. The other answer is that the transformation from a dynamic to a quasi-static problem does not always work as easily as in case of Problem #3.

Regarding the FOC of Problem #3, it is sufficient for our purposes to examine a more restricted class of utility functions, time-additive utility. Time-additive utility is defined as

\[
u(c_1, \ldots, c_T) = \sum_{t=0}^{T} \beta^{t} \cdot U(c_t)
\]

where \( 0 < \beta < 1 \) is the individual time-discount factor. The discount factor can also be written in terms of the rate of time preference \( \tau = 1/\beta - 1 \) as \( \beta = 1/(1+\tau) \). [Warning: Don’t confuse \( r \) and \( \tau \)! Time preference is a matter of personal preferences, potentially heterogeneous across agents. The interest rate characterizes individuals trading opportunities on financial markets; it affects individuals constraints, but not preferences.] The Lagrangian problem can then be written as

Maximize \( L = \sum_{t=0}^{T} \beta^{t} \cdot U(c_t) + \sum_{t=0}^{T} \lambda_t \cdot [y_t - c_t + (1+r_t) \cdot b_{t-1} - b_t] \)

where \( \lambda_t \) are the multipliers for each period \( t=0, \ldots, T \).

The first order conditions with respect to \( c_t \) are

\[
(8a) \quad \beta^{t} \cdot U'(c_t) - \lambda_t = 0, \quad t=0, \ldots, T
\]

since \( c_t \) appears in the period-\( t \) objective and in the period-\( t \) constraint. For \( b_t \), it is useful to write out the set of Lagrange terms as
\[ + \lambda_0 \cdot [y_0 - c_0 - b_0] \]
\[ + \lambda_1 \cdot [y_1 - c_1 + (1+r_1) \cdot b_0 - b_1] \]
\[ + \ldots \]
\[ + \lambda_{T-1} \cdot [y_{T-1} - c_{T-1} + (1+r_1) \cdot b_{T-2} - b_{T-1}] \]
\[ + \lambda_T \cdot [y_T - c_T + (1+r_T) \cdot b_{T-1}], \]

to see that \( b_t \) appears negatively in the period-\( t \) constraint and positively, multiplied by \( (1+r_{t+1}) \), in the period-\( t+1 \) constraint. Hence, the FOC with respect to \( b_t \) are

\[(8b) \quad (1+r_{t+1}) \cdot \lambda_{t+1} - \lambda_t = 0, \quad t=0,\ldots,T-1 \]

The multiplier \( \lambda_t \) can be interpreted as the shadow value of resources in period \( t \). According to \( (8a) \), individuals consume until marginal utility equals the shadow value of resources. For concave utility, \( U'(\cdot) \) is a decreasing function of \( c_t \), so that a low shadow value implies high consumption, and vice versa. According to \( (8b) \), it is optimal to save in period \( t \) (reduce \( c_t \)) until a real dollar in period \( t \) has the same utility value as \( (1+r_{t+1}) \) real dollars in period \( t+1 \).

Regarding complications, two examples may suffice. First, consider credit constraints: Suppose consumers cannot borrow more than the amount \( B \), so that the credit constraints \( b_t \geq -B \) apply for all \( t \). Such constraints can easily be added to Problem#3 as Kuhn-Tucker constraints; but substituting away the bond holdings would be a mess—not worth doing. Similar problem arise if loan rates and deposit interest rates differ. Second, consider a production problem: To save in period \( t \), individuals invest in capital \( k_{t+1} \). In period \( t+1 \), capital is used to produce goods \( y_{t+1} = f(k_{t+1}) \), where \( f(\cdot) \) is an increasing and concave production function. The budget equation in period \( t \) would be \( k_{t+1} = f(k_t) - c_t \), replacing \( (5) \) in Problem#3. We will examine this problem in more detail later. The marginal return to capital \( f'(k) \) could still be interpreted as an interest factor, but the interest rate would no longer be exogenous; hence, one would not gain much by writing the problem in a present value form.

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2 Equation \( (8a) \) reveals why this approach is most useful for time-additive utility. Here, \( \lambda_t \) depends only on \( c_t \). For other utility functions, marginal utility depends on consumption in all periods. Then the FOC \( u_t(c_0,\ldots,c_T) = \lambda_t \) are much less insightful.
Solution Techniques:

For now, we will stick to Problem #3 in unmodified form and examine different ways in which it can be solved. For more complicated problems, one or more of the same approaches will usually work. The principal techniques are:

(a) The Lagrange multiplier method (Applied to #2 or #3)
(b) Substituting the constraints into the objective function and then solving an unconstrained problem.
(c) Dynamic Programming.

The Lagrange multiplier methods were discussed above.

Next, consider substituting the constraints into the objective function. The idea is to write (5) as

\[ c_t = y_t - b_t + (1+r_t) \cdot b_{t-1} \]

and to substitute into the objective. Then the problem reduces to

Problem #4:

Maximize \[ u = u(y_0 - b_0, y_1 - b_1 + (1+r_1)b_0, \ldots, y_{T-1} - b_{T-1} + (1+r_{T-1})b_{T-2}, y_T + (1+r_T)b_{T-1}), \]

over choice variables \( b_0, \ldots, b_{T-1} \), i.e., an unconstrained problem with \( T \) variables. This is again a problem that looks easier than Problem #3. One a solution for \( b_0, \ldots, b_{T-1} \) is obtained, the implied consumption path can easily be computed from (9).

To compare the different solution techniques, it is instructive to compare the structure of the FOC for the case of time-additive utility. For problem #3, this was done above. With time-additive utility, Problem #2 calls for maximizing the Lagrangian

\[ L = \sum_{t=0}^{T} \beta^t \cdot U(c_t) + \Lambda \cdot [\sum_{t=0}^{T} p_t \cdot y_t - \sum_{t=0}^{T} p_t \cdot c_t], \]

which yields first order conditions:

\[ \beta^t \cdot U'(c_t) = p_t \cdot \Lambda. \]

Comparing (8a) and (10), one can see that the \( \lambda_t \) constraints can be interpreted as the period-\( t \) values of the period-0 shadow value \( \Lambda \), \( \lambda_t = p_t \cdot \Lambda \).

In Problem #4, each \( b_t \) appears twice in the objective function so that the FOC are

\[ U'(y_t - b_t + (1+r_t)b_{t-1}) = (1+r_{t+1}) \cdot \beta \cdot U'(y_{t+1} - b_{t+1} + (1+r_t)b_t) \]
for $t=0,\ldots,T-1$; they form a system of $T$ equations in the $T$ choice variables.

If one eliminates the multipliers from (8a-b) and (10) and inserts (9) into (11), the FOC for all three problems (#2, #3, and #4) reduce to the same optimality conditions for consumption:

$\frac{\beta \cdot U'(c_{t+1})}{U'(c_t)} = \frac{p_{t+1}}{p_t} = \rho_{t+1} = \frac{1}{1+r_{t+1}}$, for $t=0,\ldots,T-1$.

That is, the marginal rate of substitution of consumption between adjacent periods must equal the discount factor (interest rate).

For small $T$, these FOC and the various constraints provide enough conditions that one can compute the optimal solution either algebraically or numerically.

For larger $T$, it is worth emphasizing that we are dealing with difference equations and that the different problems lead to different types of difference equations.

(a) Equation (12) represents a first-order non-linear difference equation for consumption. For $t=0,\ldots,T-1$, this provides $T$ equations for the $T+1$ unknown consumption values. For any “guess” of $c_0$, a sequence $c_{t+1} = (U')^{-1}\left(\frac{U'(c_t)}{(1+r_{t+1}) \cdot \beta}\right)$ that pins down the “trajectory” of the consumption path. The starting value $c_0$ (the boundary condition) can be obtained by inserting the alternative sequences into the IBC. For example, consider $U(c_t) = \ln(c_t)$, $r_t = r$, $y_t = y$ for all $t$. Then the FOC $1/c_t = (1+r_{t+1}) \cdot \beta \cdot 1/c_{t+1}$ imply the difference equation $c_{t+1} = (1+r) \cdot \beta \cdot c_t$. Inserting $c_t = (1+r)^t \cdot \beta ^t \cdot c_0$ and $p_t = 1/(1+r)^t$ into the IBC implies

$$\sum_{t=0}^{T} p_t \cdot c_t = \sum_{t=1}^{T} \frac{(1+r)^t \cdot \beta ^t \cdot c_0}{(1+r)^t} = \sum_{t=0}^{T} \beta ^{t-1} \cdot c_0 = \frac{1-\beta ^{t+1}}{1-\beta } \cdot c_0$$

$$\sum_{t=0}^{T} p_t \cdot y_t = \sum_{t=1}^{T} \rho ^t \cdot y = \frac{1-\rho ^{T+1}}{1-\rho} \cdot y$$

$$\Rightarrow c_0 = \frac{1-\beta}{1-\beta ^{t+1}} \cdot \frac{1-\rho ^{T+1}}{1-\rho} \cdot y$$

(b) Equation (11) represents a second-order difference equation for bond holdings. This formulation will be more useful later when we deal with capital, because (11) is often algebraically messy. The general point is that the optimal solution can be characterized in terms of “state variables” that characterize initial conditions in a period; here, $(1+r_t) \cdot b_{t-1}$. In general, a
second-order difference equation needs two boundary conditions. Here, we have \( b_T = b_{-1} = 0 \), the initial and the endpoint conditions.

(c) Equations (8a), (8b), and (5) represent a system of first-order difference equations for bond holdings, consumption and the Lagrange multipliers:

\[
\begin{align*}
    b_t - b_{t-1} &= y_t - c_t + r_t \cdot b_{t-1} & \text{for } t = 0, \ldots, T \\
    \lambda_{t+1} - \lambda_t &= -\frac{r_{t+1}}{1+r_{t+1}} \cdot \lambda_t & \text{for } t = 0, \ldots, T-1 \\
    \lambda_t &= \beta^t \cdot U'(c_t) & \text{for } t = 0, \ldots, T
\end{align*}
\]

with boundary conditions \( b_{-1} = b_T = 0 \). This looks quite complicated here, but turns out to be very convenient in the limit when the time interval becomes small and is known as the "Maximum principle" (see Dixit, ch.10).

Dynamic programming will be discussed later.

Infinite Horizon Problems

Actual economies do not end. In macroeconomics, we are therefore often interested in the case \( T \to \infty \). For preferences, time additivity is again convenient to deal with the limit case. The objective function is assumed to be

\[
u = \sum_{t=0}^{\infty} \beta^t \cdot U(c_t),
\]

provided the sum converges. The main question is the how to write the constraints. A very direct approach is to work with the IBC. Assuming that the relevant sums in the IBC (3) converge, maximize utility subject to

\[
\sum_{t=0}^{\infty} p_t \cdot c_t = \sum_{t=0}^{\infty} p_t \cdot y_t
\]

The first order conditions are again \( \beta^t \cdot U'(c_t) = p_t \cdot \Lambda \), where \( \Lambda = U'(c_0) \). This is a system with an infinite number of equations, but sometime easy to solve, e.g., in the log-utility case.

Alternatively, one may want work with the budget equations (5). Then the main question is what to do about the terminal condition \( b_T \geq 0 \). For any finite \( T \), recall that Problems #2 and #3 are equivalent, if and only if \( p_T \cdot b_T = 0 \). In the limit, the problems are therefore equivalent if and only if

\[
\lim_{T \to \infty} p_T \cdot b_T = 0
\]

is satisfied, the so-called the transversality condition. Note that for \( r_T = r > 0 \), \( p_T = (1+r)^{-T} \to 0 \) for \( T \to \infty \). Hence, (14) allows asset position to grow.
exponentially, provided the rate of growth is less than the interest rate. The first order conditions
\[ \beta^t \cdot U'(c_t) = \lambda_t, \]
and
\[ (1+r_{t+1}) \cdot \lambda_{t+1} = \lambda_t, \]
or equivalently,
\[ U'(c_t) = (1+r_{t+1}) \cdot \beta \cdot U'(c_{t+1}) \]
again form a set of difference equations that are similar as in the case of finite horizons, except that the boundary conditions are different: \( b_T = 0 \) is replaced by (14), and/or (3) is replaced by (13).

**Note on Labor-Leisure Choices**

The above analysis assumed individuals care about only one commodity per period, consumption \( c \). Additional commodities can be added easily, for example leisure (or later, public goods). Labor-leisure choices deserve comment because they are important for business cycle.

Let time be normalized to one, \( n_t \) denote labor supply, \( 1-n_t \) denote leisure, and let preferences be
\[ u = \sum_{t=0}^{T} \beta^t \cdot U(c_t, 1-n_t) \]
where \( U \) is increasing in both arguments. Let \( w_t \) be the wage rate, then
\[ b_t - b_{t-1} = w_t \cdot n_t - c_t + r_t \cdot b_{t-1} \quad \text{for } t=0, \ldots, T \]
are the constraints. The FOC for optimal labor supply, consumption, and bond holdings are
\[ \beta^t \cdot U_{1-n}(c_t, 1-n_t) = w_t \cdot \lambda_t, \]
\[ \beta^t \cdot U_c(c_t, 1-n_t) = \lambda_t, \]
and
\[ (1+r_{t+1}) \cdot \lambda_{t+1} = \lambda_t. \]
The latter two conditions are as before; the new FOC for labor supply provides one additional equation to pin down one additional choice variable.