Risk Management for Financial Institutions

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Introduction

The Practical Problem: Regulatory Capital

- choice between “standard method” and “internal model method” since 1998

Value at Risk = estimated quantile of the distribution of the portfolio’s loss (over the next 10 days at the 99%-level)

\[
capital_t \geq \max(f \cdot \frac{1}{60} \sum_{i=1}^{60} r_{t-i}, r_{t-1})
\]

\( f = 3 + f_1 + f_2 \) : “hysteria factor”

0 ≤ \( f_1, f_2 \leq 1 \) : additional loadings for qualitative and quantitative deficiencies

\( r_t \): Value at Risk reported at \( t \)

(BAKred “Grundsatz I” 2000)

incentives:

1. lower required capital allows more business / higher expected returns
2. too low estimates of VaR can increase the factor \( f_1 \) (back-testing)

input: portfolio positions, contract specifications, and historical data for some 100000 securities
• commodities (precious metals, oil, . . .)
• stocks
• fixed income, FX (depends on maturity and counter-party)
• futures (depends on maturity)
• options (depends on strike, maturity, underlying, . . .)

aspects:
1. IT infrastructure (CORBA, database technology)
2. statistical modeling (dimension reduction, stochastic volatility)
3. computation (integration in high dimensions, approximation of distributions)

Why Important?

internal models:
1. lower capital requirements
2. approval by the BAKred can improve rating, standing, or volume
3. no need for separate internal and external control systems; the alternative (“standard model”) is too inflexible for derivatives
other uses of risk measures:

1. information reporting (shareholders, senior management)
2. performance measurement, compensation
3. resource allocation, controlling

for mathematicians:

- bigger market than the valuation and hedging of derivatives (anecdotal evidence)

Skeleton

1. The Need for Risk Management - Practical Problems
2. VaR and Coherent Risk Measures
3. VaR and Expected Shortfall for Portfolios
   i) independent multivariate normal risk factors (RiskMetrics)
   ii) independent identically distributed (iid) risk factor changes (historical simulation)
   iii) stochastic volatility models
   iv) (linear) dimension reduction techniques
   v) separate modeling of marginal distributions and dependence structure: correlations and copulas
Where is the Math?

A. information based complexity (Traub and Werschulz; 1998) and applications to integration in high dimensions

B. extreme value theory (Embrechts et al.; 1997)

IBC Example

How many function evaluations are needed to achieve a certain level of accuracy $\epsilon$?

Multivariate integration is intractable in the worst-case setting:

$$\text{complexity}_{\text{worst}}(\epsilon, d) \sim c(f, d)\epsilon^{-d/r}$$

(Bakhvalov 1959)

$c(f, d)$: cost of function evaluation

$\epsilon$: approximation accuracy

d: dimension

$r$: derivatives up to degree $r$ are bounded

- similar in spirit to complexity theory based on Turing-machine: provides lower bounds; classifies
problems
- very important alternative complexity theory

**EVT Example**

\[ X_1, \ldots, X_n \text{ iid } \sim F \]

\[
\frac{\max\{X_1, \ldots, X_n\} - a_n}{b_n} \xrightarrow{d} H
\]

- only *extreme value distributions* (Gumbel, Weibull, Frechet) can appear as the limit \( H \)
- virtually all known distributions are in the *maximum domain of attraction* of one of the extreme value distributions

(Fisher-Tippett 1928)
- long-standing applications in hydrology

**Part II**

6. Representations of Partial Preferences
7. Valuation Bounds in Incomplete Markets
8. Risk-constrained Portfolio Optimization

**Portfolio Optimization**

standard: Markowitz, CAPM, based on $\mu, \sigma$

alternative: $(\mu, \rho)$-optimization for general risk measures $\rho$:

$$\max_{x-x_0 \in M} \{\mu(x) \mid \rho(x) \leq c\}$$

*(Markowitz; 1959)*

**Valuation and Hedging of Derivatives**

standard: Black/Scholes, complete market models

alternative: hedging is special case of PO-problem

$\rightarrow$ optimal hedge & price bounds depend on: initial position, transaction costs, trading constraints, ...

*(Hodges and Neuberger; 1989)*

! important to first know how to *measure* risk before formulating optimization problems based on risk measures

**guess:** VaR-infrastructure/IT-investments allow to actually implement some of the incomplete market models
The Need for Risk Management

The Historic View

sudden increase in financial market risks

1971 breakdown of the fixed exchange rate system

1973 oil price shock, volatile interest rates

financial derivatives

1972 FX futures traded in the “International Monetary Market” (IMM) at the Chicago Mercantile Exchange (CME)

1973 stock options traded at the Chicago Board Options Exchange (CBOE), founded by members of the Chicago Board of Trade (CBOT)

1975 GNMA futures (CBOT)

1977 Treasury bond futures (CBOT)

1982 options on T-bond futures (CBOT)
1983 stock index options (CBOE)
(Further events see (Jorion; 2000, Table 1-2, p.13).)

option pricing theory
1973 Black and Scholes (geometric Brownian motion, based on PDE arguments)
1976 Cox and Ross (alternative processes, probabilistic arguments)
1979 Harrison and Kreps (“nice” martingale-theoretic understanding)
1987 stock market crash
   breakdown of portfolio insurance
   pronounced volatility smile ever since

regulatory action

Jul 1988 Basel Accord (BCBS88)
   goal: secure bank deposits, control of credit risks
   “capital” $\geq$ 8% of “risk-weighted assets”
   restriction of “large risk”
   $\rightarrow$ static, diversification ignored
1989 EU “Solvency Ratio” and “Own Funds” Directives

1991 Federal Deposit Insurance Corporation Improvement Act

Apr 1993 (BCBS93): first proposal of the “building-block approach”, now called “standard method”, superseded by (BCBS95)

- goal: control of market risks
- decomposition: commodity risk, FX rate risk, interest rate risk, equity risk
- “capital” ≥ 8% of “risk-weighted assets”
  → still static, some diversification


the disaster period

Feb 1993 Showa Shell $1580 in currency forwards

Jan 1994 Metallgesellschaft $1340 in oil futures

Apr 1994 Kashima Oil $1450 in currency forwards

Dec 1994 Orange County $1810 in reverse repos

Feb 1995 Barings $1330 in stock index futures
(further losses and stories in (Jorion; 2000, chapter 2))

**flurry of private sector and regulatory activity**


**May 1994** General Accounting Office report


**Oct 1994** JP Morgan: RiskMetrics

**regulatory action**

**Apr 1995** *(BCBS95)*: proposal for *internal models*

**Jan 1996** *(BCBS96)*: Amendment to the Basel Accord (internal models) (effective January 1998)


**Jun 1998** CAD 2 *(EU Council; 1998a)*

**Jul 2000** BAKred: “Grundsatz I” *(Bundesaufsichtsamt für Kreditwesen; 2000)*
Current Rules

Grundsatz I, §33:
internal model covers “general market risks” only or includes also “specific event risk”:

\[ MRC_t = \max(f \cdot \frac{1}{60} \sum_{i=1}^{60} r_{t-i}, r_{t-1}) \]

internal models covers “specific market risks”, but not “event risk”:

\[ MRC_t = \max(f \cdot \frac{1}{60} \sum_{i=1}^{60} r_{t-i} + \frac{1}{60} \sum_{i=1}^{60} s_{t-i}, r_{t-1} + s_{t-1}) \]

\((s_t\) is the “specific risk” surcharge at \(t\).)

- written approval by BAKred (§32)
- 10 days horizon, 99% probability level, based on at least 1 year’s historic data (§34)
- “sufficient” set of risk factors: risks associated with nonlinear instruments, term structure of interest rates and spread risks, spot-forward spreads for commodities (§35)
- qualitative requirements (§36)
- backtesting (§37): daily VaR is compared to actual exceedences in the last 250 days (“clean P&L”) add-on factor \(f_1:\)
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<th>factor</th>
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<td>9</td>
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<tr>
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<td>≥ 10</td>
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</tr>
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**Outlook**

**consequences for**
- portfolio optimization
- hedging and valuation of derivatives
- incomplete market models

**not yet implemented in law**
1995  Fed: pre-commitment approach (Kupiec and O’Brien)
   no need for “clean P&L” computations

   “three pillars”:
   1. minimum capital requirements (external credit assessments)
   2. a supervisory review process
   3. effective use of market discipline
   “maybe”: internal credit ratings and portfolio credit risk modeling


Jan 2000  (BCBS2000a): pillar 3, more disclosure

Sep 2000  (BCBS2000c): credit risk modeling

**Suggested Homework**

Skim the first 6 sections of the “Grundsatz I” (Bundesaufsichtsamt für Kreditwesen; 2000) to get an overview of the “standard method” (and appreciate the alternative “internal model method”). Read section 7 containing the current implementation of the “internal model method” in German law.
Further Reading

A good overview of the developments leading to regulatory Value at Risk is given by (Jorion; 2000, chapters 1-3). Chicago was the birth place for exchange traded derivatives, see the web sites Chicago Board of Trade (2000), Chicago Mercantile Exchange (2000a), and Chicago Board Options Exchange (2000). Davies has a nice collection of web links to documents about financial scandals. The texts of the Basel Committee on Banking Supervision are available from http://www.bis.org/publ/pub_list.htm. (Basel Committee on Banking Supervision; 2000a) gives a short historic view of the Committee’s activities and main proposals. The EU consultation documents are available from http://europa.eu.int/comm/internal_market/en/finances/banks/, directives from http://europa.eu.int/eur-lex/. Most German legal texts regarding banking regulation are available from http://www.bakred.de. Traber (2000) provides insight into the process of the approval of internal models from the viewpoint of a regulator.
VaR and Other Risk Measures

VaR

usage of the term *Value at Risk*:

1. VaR is an estimate of some upper bound on the likely loss of a portfolio’s market value over a target horizon.

   \[ \rho(-P&L(\phi)) \]

2. VaR is the estimated quantile of the portfolio’s loss over a target horizon.

   \[ \text{VaR}_\alpha(\phi) = q_\alpha(-P&L(\phi)) \]

   i) \( q_\alpha \) is an \( \alpha \)-quantile of the distribution of a random variable \( X \) (under the probability \( P \)) if

   \[ P\{X < q_\alpha\} \leq \alpha \leq P\{X \leq q_\alpha\}. \]

   ! Quantiles for a fixed alpha form a closed interval.

   ii) “VaR summarizes the expected maximum loss over a target horizon within a given confidence interval.” (Jorion; 1997, p.19)

   \[ q^-_\alpha(X) = \inf\{x \mid P\{X \leq x\} \geq \alpha\} \]
3. VaR is the methodology of assuming normally distributed driving factors and expressing the portfolio value as a function of these underlying factors (as in RiskMetrics). (“...since VaR accounts for correlations...” (Jorion; 1997, p.285). “The other characteristic of VaR is that it takes account of correlations...” (Dowd; 1998, p.20).)

**confidence level:**
- 99% (G I)
- 95% (RiskMetrics)

**horizon:**
- 1 day (G I: for backtesting; RiskMetrics: for trading)
- 10 days (G I: for required capital)
- 25 days (RiskMetrics: for investment)

**advantages:**
- simple
- single number, can aggregate different kinds of risk, allows enterprise-wide risk management (ERM)
- in $, translation property (unlike $) 
  \[ \rho(X + \alpha 1) = \rho(X) - \alpha, \quad \forall \alpha \geq 0. \]
VaR Is the Wrong Risk Measure . . .

... for trading limits and the allocation of resources

(1) VaR is not compatible with diversification.
   • defaultable bonds (Artzner et al.; 1998, p.14, attributed to Albanese)

... for compensation

(2) VaR induces the wrong incentives in reasonably complete markets.
   • buy defaultable bonds, sell riskless bonds (LTCM)
   • sell far-out-of-the-money puts
   • doubling strategy
   • delta-hedged OTC book: short gamma, short vega
   • Peso problem traders (Taleb; 2000)

... for regulatory purposes

(3) VaR is the minimal loss of the \((100 - \alpha)\)% “bad” cases. It says nothing about the average loss in the “bad” cases.
   • OK from the viewpoint of management: interested in a long life of the firm. (Similar for shareholders.)
Not OK from the viewpoint of the central bank and the taxpayer: how much does the cleanup cost (on average)?

**alternative:** *tail conditional expectation* (alias “tail VaR”, “conditional VaR”, or “beyond VaR”):

\[
\text{TCE}_\alpha(\phi) = \mathbb{E}[-\text{P&L}(\phi) \mid -\text{P&L}(\phi) \geq \text{VaR}_\alpha(\phi)],
\]

which is related to *expected shortfall* (alias “mean excess loss”)

\[
\text{ES}_\alpha(\phi) = \mathbb{E}[-\text{P&L}(\phi) - \text{VaR}_\alpha(\phi) \mid -\text{P&L}(\phi) \geq \text{VaR}_\alpha(\phi)]
\]

by

\[
\text{TCE}_\alpha = \text{VaR}_\alpha + \text{ES}_\alpha.
\]

**Life Expectancy as a Function of the VaR Level**

**life time:**

\[
\tau = \min\{t \mid c + \sum_{i=1}^{t} X_i \leq 0\}
\]

\(X_i\) iid

\(c\) initial capital
→ “ruin theory”, “risk theory” (Cramer, Lundberg)

- not stationary, not realistic

**alternative:** “take more business as we grow, stay at the VaR-limit”:

\[
\tau = \min \{ t \mid c + X_t \leq 0 \}
\]

\((P\{-X_t \geq c\} = \alpha\) is the VaR confidence level.\)

\(\tau\) has a geometric distribution:

\[
P\{\tau = n\} = \alpha^{n-1}(1 - \alpha)
\]

\(\rightarrow E[\tau] = \frac{1}{1 - \alpha}\)

99% VaR, 10 days horizon, life expectancy in years:

<table>
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<tr>
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<th>normal</th>
<th>exponential</th>
<th>power</th>
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</tr>
<tr>
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<td>36</td>
</tr>
<tr>
<td>4</td>
<td>6.0 \cdot 10^{18}</td>
<td>4 \cdot 10^6</td>
<td>64</td>
</tr>
<tr>
<td>5</td>
<td>2.8 \cdot 10^{29}</td>
<td>4 \cdot 10^8</td>
<td>100</td>
</tr>
</tbody>
</table>

(Tails are (1) standard normal, (2) exponential \((e^x 1_{[-\infty,0]}\)), (3) power \((2|x|^{-3} 1_{[-\infty,-1]}\)).

**Conclusion**

VaR is blind towards large losses with small probabilities. This is appropriate in situations where only a small ruin probability – alias a long life expectancy – is desired, i.e. the costs associated with the ruin are essentially fixed and do not depend on the size of the exceedence. This is the case for the overflowing of
dikes. It is also approximately the case for bankruptcies from the viewpoint of management (=costs of looking for a new job) as well as the viewpoint of shareholders (=liquidation costs). It is markedly not the case for bankruptcies of banks from the viewpoint of depositors, creditors, regulatory authorities, tax payers, or whoever has to bear the loss that exceeds the capital of the failed bank.

### Computation of VaR and ES for specific distributions

\[
\text{VaR}_{\alpha}(X) = - \sup \{q \mid E[\mathbf{1}\{X \leq q\}] \leq 1 - \alpha \}
\]

“largest strike price such that the price of a digital put option is at most \(1 - \alpha\)”

\[
\text{ES}_{\alpha}(X) = \frac{1}{1 - \alpha} E[(X - \text{VaR}_{\alpha}(X))^+]\]

“price of a put option with strike at the VaR-level divided by the price of the corresponding digital option”

(In a risk-neutral world.)

Option valuation techniques can be applied to risk measurement and vice versa. Where risk measurement techniques are especially suited to measure tail behavior, they are also especially suited to value out-of-the-money options.

### 1 Standard Normal Distribution
\[ \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \]

\[ \text{VaR}_\alpha = q_\alpha \]
\[ \text{TCE}_\alpha = \frac{1}{1-\alpha} \int_{-\infty}^{q_{1-\alpha}} -x\phi(x)\,dx \]
\[ = \frac{\phi(q_{1-\alpha})}{1-\alpha} \]

\[ (\phi'(x) = -x\phi(x).) \]

2 Exponential Tails

\[ f(x) = \lambda e^{\lambda x} 1_{(-\infty,0)} \]
\[ F(x) = \max(e^{\lambda x}, 1) \]

(\( \lambda > 0. \))

\[ \text{VaR}_\alpha = -\frac{1}{\lambda} \log(1 - \alpha) \]
\[ \text{TCE}_\alpha = \frac{1}{\lambda} + \text{VaR}_\alpha \]
→ constant expected shortfall = absolute difference between VaR and TCE

3 Power Tails

\[ f(x) = \beta (-x)^{-\beta - 1} 1_{(-\infty, -1)} \]
\[ F(x) = \max((-x)^{-\beta}, 1) \]

(\(\beta > 1\).)

\[ \text{VaR}_\alpha = (1 - \alpha)^{-1/\beta} \]
\[ \text{TCE}_\alpha = \frac{\beta}{\beta - 1} \text{VaR}_\alpha \]

→ constant relative difference between VaR and TCE
$ES(a) = TCE(a) - VaR(a)$
The diagram shows the relationship between probability level $a$ and the ratio $\frac{TCE(a)}{VaR(a)}$ for different distributions:

- **Normal**
- **Exponential**
- **Power**
But: Any level of TCE can be reached by “Peso problem strategies” with restricted VaR:

\[
P\&L = \begin{cases} 
1 & \text{with probability 0.5} \\
-1 & 0.5 - p \\
-x & p 
\end{cases}
\]

with \( p < 1 - \alpha < 0.5 \) and \( x > 1 \). Then the VaR is constant

\[\text{VaR}_\alpha = 1\]

but the TCE is unbounded:

\[\text{TCE}_\alpha = 1 + 2(x - 1)p.\]

### Coherent Risk Measures

**Artzner et al. (1998):** What are desirable properties of risk measures (for regulatory purposes)?

\( \Omega \) finite. Space of random variables: \( L = \{X| X : \Omega \to \mathbb{R}\} \). Risk-less return: \( 1(\omega) \equiv 1 + r \).

- A *risk measure* \( \rho : L \to \mathbb{R} \) is called *coherent* if it has the following four properties:

  1. translation invariance

    \[\rho(x + \alpha 1) = \rho(x) - \alpha\]
+ VaR 
− not: \( \sigma \), entropy

2. convexity

\[ \rho(\lambda x + (1 - \lambda)y) \leq \lambda \rho(x) + (1 - \lambda)\rho(y) \]

= consistent with diversification and risk aversion

3. positive homogeneity

\[ \rho(\alpha x) = \alpha \rho(x), \alpha > 0 \]

− contradicts usual way of thinking about risk aversion
+ OK from viewpoint of regulators

4. monotonicity

\[ x \in L^+ \implies \rho(x) \leq 0 \]

\((\alpha \in \mathbb{R}; x, y \in L; \lambda \in [0, 1]).\)

**One-to-one Correspondences**

informally:

Modulo technical conditions, there is a one-to-one correspondence between the following economic objects:
1. “coherent risk measures” $\rho$,

2. cones of “acceptable risks” $(A = \{x \mid \rho(x) \leq 0\})$,

3. partial orderings “$x \succeq y$”, meaning “$x$ is at least as good as $y$” ($x \succeq y \iff \rho(x - y) \leq 0$),

4. valuation bounds $\pi$ and $-\pi$ (with $\rho(x) = \pi(-x) = -\pi(x)$), and

5. sets $K$ of “admissible” price systems $\pi$ ($\pi \in K \iff \pi(x) \geq 0$ for all $x \succeq 0$).

▶ **coherent acceptance set:**

   (a) $A$ is a cone,

   (C) $A$ is closed and

   (M) $L^+ \subset A$.

▶ **coherent partial preferences:**

   (a) $\succeq$ is a vector ordering

   (C) $\{x \mid x \succeq y\}$ is closed, and

   (M) $x \geq 0 \implies x \succeq 0$

▶ **coherent valuation bounds:**

$\pi(x) = -\pi(-x)$ and $-\pi$ is a coherent risk measure.
A coherent set of admissible price systems:

(a) \( K \) is a cone,
(b) \( K \) is closed, and
(c) \( x \in L^+ \implies \pi(x) \geq 0 \) for all \( \pi \in K \).

formally:

There is a one-to-one correspondence between coherent risk measures, coherent acceptance sets, coherent partial preferences, coherent valuation bounds, and coherent sets of admissible price systems.

A normalized price system \( \pi \) is a linear mapping from \( L \) to \( \mathbb{R} \) with \( \pi(1) = 1 \) and \( x \geq 0 \implies \pi(x) \geq 0 \).

**Representation Theorem**

There is a one-to-one correspondence between normalized, non-negative price systems and probability measures \( Q \) by the mappings

\[
\pi(X) = \mathbb{E}_P[x/(1+r)] \quad \text{and} \\
P(A) = \pi(1_A),
\]

where \( 1_A = (1+r)\chi_A \) and \( \chi_A \) is the indicator function of \( A \).

**strong duality theorem:**
Let $A$ be a closed cone containing $1$, $K = A^\top$ its polar cone, and

$$(P) \quad \rho(x) = \inf \{p \mid p1 + x \in A\}$$

the associated risk measure.

(i) The set of normalized admissible price systems

$$D := \{\pi \in K \mid \pi(1) = 1\}$$

is not empty iff $-1 \not\in A$ ($\rho > -\infty$).

(ii) In this case

$$(D) \quad \rho(X) = \sup_{\pi \in D} -\pi(X).$$

Analogously:

$$\bar{\pi}(X) = \sup_{\pi \in D} \pi(X)$$

$$\underline{\pi}(X) = \inf_{\pi \in D} \pi(X)$$

→ "extended present value principle"

Using the one-to-one correspondence between normalized price systems and probability measures, the duality theorem implies the

! representation theorem:
\( \rho \) is a c.r.m. iff there exists a set \( Q \) of probability measures:

\[
\rho(X) = \sup_{Q \in \mathcal{Q}} -E_Q[X/(1 + r)]
\]

**Examples**

Chicago Mercantile Exchange (2000b):
- based on scenarios, of the form \( \sup_{Q \in \mathcal{Q}} -E_Q[X] \)

The SEC rules:
- benchmark securities \( Y_i \)
- assign risk numbers \( \rho_i \)
  - \( A := \text{cone}\{Y_i + \rho_i \mathbf{1}\} + L^+ \)
  - \( A \) is coherent

**Worst conditional expectation:**

\[
\text{WCE}_\alpha(X) := \sup_{B: P(B) \geq 1-\alpha} E_P[-X/(1 + r) | B].
\]

\( \text{TCE}_\alpha \) is a “solution” to this optimization problem (exact in special cases, approximation otherwise, see (Artzner et al.; 1998)).
General Risk Measures

- **partial preferences** \(\succeq\):
  1. \(\succeq\) is a pre-order,
  2. the level sets \(\{y \mid y \succeq x\}\) are convex,
  3. \(\succeq\) is monotone: \(x \geq y\) implies \(x \succeq y\).

- associated **acceptance set** \(A(e)\) (depending on the initial endowment \(e\)):
  \[
  A(e) = \{x \mid x + e \succeq e\}.
  \]

- associated **risk measure** \(\rho(., e)\):
  \[
  \rho(x, e) = \inf_{p \in \mathbb{R}} \{p \mid p \mathbf{1} + x + e \succeq e\}
  \]

**VaR is the Industry Standard**

- legal texts (G I)
- RiskMetrics is the benchmark
- widely implemented

ES may be used as a complement to VaR in the future ([Härdle and Stahl; 1999](#)).
Further Reading

Artzner et al. (1998) introduce coherent risk measures and provide lots of economic arguments. Jaschke and Küchler (1999) generalize the concept to linear spaces and add the relation between valuation bounds and risk measures. (Jaschke and Küchler; 1999) also contains some examples of risk measures and partial orderings that are not coherent.

Nassim Taleb is well-known for his incisive critique of (the usual/naive application of) Value at Risk. Many important insights into risk management in general are contained in the first (online) sections of (Taleb; 2000). It contains some real life examples of traders who continuously earned several millions per year for their firms over many years and hence were very highly valued. Then they lost a multiple of what they had earned over the previous years in a few weeks. (Needless to say that they lost their jobs.) As a matter of fact, there are trading strategies that produce long stretches of low-volatility excess returns, but then crash big. Because buying emerging market bonds is one way to achieve such behavior Taleb calls traders running such strategies “Peso problem traders”. Now the point is to observe that Value at Risk (in both the narrower and wider senses) fails to “see” the risk in these “Peso problem strategies”.

VaR and TCE for Portfolios

The Usual Classification

Longerstaey (1996); Jorion (1997); Dowd (1998):

1. “Variance-Covariance Approach”, “Delta- (Gamma-) Normal”, “Analytic Method”
2. “Historical Simulation”
3. “Structured Monte Carlo”

Variance-Covariance Approach

modeling decision:

\[ P&L_t(\phi, X) = \sum_i \phi_i f_i(X_t) \]

The innovations in the underlying risk factors \(X_t\) are iid multi-variate normal \(N(\mu, \Sigma)\). \(\mu\) is usually 0.

estimation decision: Estimate \(\Sigma\) as moving average or exponentially weighted moving average (EWMA) of empirical covariances.
approximation decision:

\[
 f_i(x) \approx f_i(0) + \sum_{j=1}^{m} x_j \frac{\partial f_i}{\partial x_j}(0) \quad \text{(Delta-Normal)}
\]

\[
 f_i(x) \approx f_i(0) + \sum_{j=1}^{m} x_j \frac{\partial f_i}{\partial x_j}(0) + \sum_{j=1}^{m} \sum_{k=1}^{m} x_j x_k \frac{\partial^2 f_i}{\partial x_j \partial x_k}(0)
\]

(Delta-Gamma-Normal)

**Computation of VaR and TCE (for Delta-Normal)**

\( (f_i(0) = 0, \mu = 0) \)

\[
 \text{P&L} \approx \sum_i \phi_i \sum_{j=1}^{m} X_j \frac{\partial f_i}{\partial x_j}(0)
\]

\[
 = \Delta^\top X
\]

with \( \Delta_j = \sum_i \phi_i \frac{\partial f_i}{\partial x_j}(0) \).

\( \Delta^\top X \sim N(0, \Delta^\top \Sigma \Delta) \)

\[\implies\]

\[
 \text{VaR}_\alpha(\phi) = q_\alpha(N(0, 1)) \cdot \sigma \\
 \text{TCE}_\alpha(\phi) = \text{TCE}_\alpha(N(0, 1)) \cdot \sigma
\]
with $\sigma = \sqrt{\Delta^\top \Delta}$.

### Historical Simulation

**modeling decision:**

$$P&L_t(\phi, X) = \sum_i \phi_i f_i(X_t)$$

The innovations in the underlying risk factors $X_t$ are iid $\sim F$, not necessarily normal.

**estimation decision:** Take the empirical distribution function of past observations $X_{t-k}$ as an estimator for $F$. Plug the historic samples into the P&L-equation and use empirical quantiles, smoothing techniques, or EVT to estimate the quantile of the P&L-distribution function.

**no approximation necessary**

### Structured Monte Carlo

**statistical and estimation decisions:** Unspecified (but in practice often just iid multi-variate normal increments).

**approximation decision:** By using pseudo random numbers or quasi random numbers generate a sample $(x_1, \ldots, x_n)$ according to the assumed distribution of $X_t$. Plug the samples into the P&L-equation and use empirical quantiles to estimate the quantile of the P&L-distribution function.
Separation of Different Aspects

Critique:

- statistical and numerical decisions not well separated
- methods are usually compared on few selected portfolios → anecdotal evidence only

<table>
<thead>
<tr>
<th>statistical</th>
<th>numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td>which model?</td>
<td>which estimator?</td>
</tr>
<tr>
<td></td>
<td>how to approximate/compute?</td>
</tr>
<tr>
<td>depends <em>essentially</em></td>
<td>given a model, depends only on the portfolio or class of possible portfolios</td>
</tr>
<tr>
<td>on the statistical properties of market data</td>
<td></td>
</tr>
</tbody>
</table>

→ Compare statistical decisions w.r.t. a model choice criterion by using exact computations on the computation side.

→ In a given model, compare approximation techniques w.r.t. worst-case or average case approximation errors over classes of portfolios.
Mapping

1. *mapping in the narrow sense*: how to choose \( f_i \) for given risk factors \( X \) (like in RiskMetrics) in

\[
P\&L(\phi) = \sum_i \phi_i f_i(X).
\]

2. *mapping in the wide sense*: how to choose \( X \) in the first place
   - dimension reduction
     - statistical decision (factor analysis, subspace methods)

**Mapping in the narrow sense** (*Longerstaey; 1996, chapter 6*)

**Step 1**: break down contracts into individual payments

- bond = portfolio of zero-bonds
- FRN = zero-bond, maturing at the next settlement date
- swap = bond - FRN
- FRA\((t_1,t_2)\) = zero-bond\((t_1)\) - zero-bond\((t_2)\)
- same for interest rate futures
- FX forward = zero-bond\((t,c_1)\) - zero-bond\((t,c_2)\)
- currency swaps = bond\((c_1)\) - FRN\((c_2)\)
- commodities: forwards and swaps like currency forwards and swaps
• options: \( f_i = \) Black-Scholes-formula; risk factors: the underlying, the implied volatility, the interest rate

→ mapping positions in contracts to positions in “building blocks”:

• single shares
• single deterministic future payments
• single future payments/delivery in foreign currencies or commodities
• implied volatilities

**Step 2:** map positions in building blocks to positions in risk factors

1. equity, FX rates, commodity prices
   i) “building block” is a risk factor itself
      i. log returns are assumed to be normally distributed:
         \[
         \begin{align*}
         \text{P&L} &= P_t + D_t - P_{t-1} \\
         r_t &= \log((P_t + D_t)/P_{t-1}) \\
         f(x) &= e^x - 1 \\
         \text{P&L} &= P_{t-1}f(r_t)
         \end{align*}
         \]
ii. percent returns are assumed to be normally distributed:

\[ R_t = \frac{(P_t + D_t - P_{t-1})}{P_{t-1}} \]

\[ f(x)) = x \]

\[ \text{P&L} = P_{t-1} f(R_t) \]

<table>
<thead>
<tr>
<th></th>
<th>percent returns</th>
<th>log returns</th>
</tr>
</thead>
<tbody>
<tr>
<td>normality OK</td>
<td>−</td>
<td>+</td>
</tr>
<tr>
<td>f linear</td>
<td>+</td>
<td>−</td>
</tr>
<tr>
<td>FX cross rates OK</td>
<td>−</td>
<td>+</td>
</tr>
</tbody>
</table>

ii) mapping to an index (CAPM-like)

\[ r_t = \beta r_{m,t} + \mu + \epsilon_t \]

\[ f(x) = e^{\beta x + \mu} - 1 \]

\[ \text{P&L} \approx P_{t-1} f(r_{m,t}) \]

(ignoring the firm-specific risk \( \epsilon_t \))

2. zero-bonds

i) exact maturity is a risk factor

i. log returns are normal
   - RiskMetrics
   - somewhat unrealistic since price \( \leq 1 \)
ii. changes in yields are log-normal:

\[ P_t(\tau) = (1 + y_t(\tau))^{-\tau} \]

\[ x_t = \log(y_{t+1}(\tau - 1)/y_t(\tau)) \sim N(0,1) \]

P&L = \[ P_{t+1}(\tau - 1) - P_t(\tau) \]

\[ = (1 + e^{x}(P_t(\tau)^{-1/\tau} - 1))^{-\tau} - P_t(\tau) \]

\[ = f(x, P_t) \]

ii) mapping to selected maturities: (1m 3m 6m 1y 2y 3y 4y 5y 7y 9y 10y 15y 20y 30y)

i. (Longerstaey; 1996):
   A. value is preserved (yields are linearly interpolated)
   B. volatility is preserved (volas are linearly interpolated)
   C. sign is preserved

   \[ \rightarrow \] unique solution of a quadratic equation

   ! problem: Map is discontinuous if correlation between vertices is low.

ii. Mina (1999): simple linear mapping:

\[ d_\tau = \alpha d_{\tau_L} + (1 - \alpha)d_{\tau_R} \]

with

\[ \alpha = \frac{\tau_R - \tau}{\tau_R - \tau_L}. \]

A. does not have the discontinuity problem of the standard RiskMetrics mapping method

iii. nonlinear mapping, e.g. yield is linearly interpolated \rightarrow \] interpolated discount factors are not
a linear function of discount factors with selected maturities

3. later delivery of commodities, implied volatilities
   • roughly similar to zero-bond-mapping

**Conclusion**

The discussion in both Longerstaey (1996) and Mina (1999) is somewhat limited as it focuses on linear mapping.

Mapping (in the narrow sense) is highly dependent on the choice of risk factors (mapping in the wider sense) and should be viewed in the context of (statistical) dimension reduction techniques.
Delta-Gamma-Methods

risk factors $X \sim N(0, \Sigma)$

$$P = \theta^\top 1 + \Delta^\top X + \frac{1}{2} X^\top \Gamma X$$

These “Greeks” (=derivatives w.r.t. the risk factors) have to be computed from the usual Greeks (=derivatives w.r.t. the underlyings) by use of the chain rule of differentiation.

Fourier Inversion

1. step: Express $P$ as sum of independent non-central $\chi_1^2$ variates.

Choose linear transformation $X = CY$ such that

(1) $Y \sim N(0, I)$

(2) $P = \theta^\top 1 + (C^\top \Delta)^\top Y + \frac{1}{2} Y^\top \Lambda Y$

for a diagonal $\Lambda$. This is a generalized symmetric eigenvalue problem:

(1) $CC^\top = \Sigma$

(2) $C^\top \Gamma C = \Lambda$

(The columns of $C$ contain the generalized eigenvectors, $\Lambda$ contains the eigenvalues.)
Packages like *Anderson et al.* (1999) contain routines directly for the generalized eigenvalue problem. Otherwise $C$ can be computed in two steps:

1. Compute a matrix $B$ with $BB^\top = \Sigma$, preferably by Cholesky decomposition.
2. Solve the (standard) symmetric eigenvalue problem for the matrix $B^\top \Gamma B$:
   \[
   Q^\top B^\top \Gamma B Q = \Lambda
   \]
   and set $C := BQ$.

$P$ is sum of independent random variables:

\[
P = \sum_i (\theta_i + \delta_i Y_i + \frac{1}{2} \lambda_i Y_i^2)
\]

\[
= \sum_i \left\{ \theta_i + \frac{1}{2} \lambda_i \left( \frac{\delta_i}{\lambda_i} + Y_i \right)^2 - \frac{\delta_i^2}{2\lambda_i} \right\}
\]

with $\delta = C^\top \Delta$.

**2.step:** compute the mgf of $P$ (= product of mgfs of non-central $\chi_1^2$ variates)

The moment generating function of a non-central $\chi_1^2$ is known analytically:

\[
E e^{t(Z + a)^2} = (1 - 2t)^{-1/2} \exp \left( \frac{a^2 t}{1 - 2t} \right) \quad (t < \frac{1}{2})
\]
This implies the mgf for $P$:

$$E e^{tP} = \prod_i \exp \left( \theta_i t + \frac{1}{2} \delta_i^2 t^2 (1 - \lambda_i t)^{-1} - \frac{1}{2} \log(1 - \lambda_i t) \right).$$

Re-expressed in terms of $\Gamma$ and $\Sigma$:

$$\log E e^{tP} = \theta^\top 1 t + \frac{1}{2} t^2 \Delta^\top \Sigma (I - \Gamma \Sigma t)^{-1} \Delta - \frac{1}{2} \text{tr}(\log(I - \Gamma \Sigma t)) \quad (1)$$

(Mina and Ulmer (1999) seem to have some errors there.)

3. step: Fourier inversion

operation count ($m$ risk factors, $N$ discretization points):

- generalized eigenvalue problem: $O(m^3)$
- mgf: $O(mN)$
- FFT: $O(N \log N)$

eigenvalue problem dominates: 3 minutes on a Pentium II 450 MHz for 500 risk factors

Summary

- This technique is more general than just Gaussian: works well if characteristic function is known.
- It works even if the characteristic function is not known but the distributions of the $Y_i$'s can be
estimated and the \( Y_i \)'s can be assumed independent. Gradients (sensitivities) of VaR w.r.t. both Delta and Gamma can also be computed by Fourier inversion (Albanese et al.; 2000).

- A Fourier inversion technique can be applied even if the \( Y_i \)'s are not independent (Glasserman et al.; 2000), but the method is then much slower.
- Only drawback: relatively high “start-up” cost of doing the eigenvalue decomposition.

### Moment-Based Methods

cumulants of the P&L-distribution are known:

\[
\kappa_r = \frac{1}{2} \sum_i \{(r - 1)! \lambda_i^r + r! \delta_i^2 \lambda_i^{r-2}\} \quad (r \geq 2)
\]

\[
\kappa_1 = \frac{1}{2} \sum_i \lambda_i
\]

and can be expressed in terms of \( \Gamma, \Sigma, \) and \( \Delta \) avoiding the eigenvalue decomposition:

\[
\kappa_r = \frac{1}{2} (r - 1)! \text{tr}((\Gamma \Sigma)^r) + \frac{1}{2} r! \Delta^\top \Sigma (\Gamma \Sigma)^{r-2} \Delta \quad (r \geq 2)
\]

### Salomon-Stephens approximation

Britton-Jones and Schaefer (1999):
approximate distribution of $P$ (or $-P$) by

$$c_1 w^{c_2} \quad (w \sim \chi^2_p)$$

$c_i$ such that first three moments are matched.

problem: distribution is bounded below (or above)

**Johnson Transformations**

Approximate $P$ by matching the first four moments with the moments of $f(X)$, where $f$ is a monotonic transformation (depending on 4 parameters) and $X \sim N(0, 1)$.

e.g.

$$f(X) = \sinh \left( \frac{X - \gamma}{\delta} \right) \lambda + \xi$$

(Moments of $f(X)$ are known explicitly.)

Mina and Ulmer (1999): difficult to fit, “not a robust choice”
Gram-Charlier Series

Ansatz for the density:

\[ f(x) = \phi(x) \sum_{k=0}^{\infty} c_k H_k(x) \]

where \( \phi \) is the standard normal density and \( H_k \) are the Hermite polynomials:

\[ \phi^{(k)}(x) = (-1)^k H_k(x) \phi(x). \]

The Hermite polynomials are orthogonal w.r.t. the weight function \( \phi \):

\[ \int_{-\infty}^{\infty} H_i(x) H_j(x) \phi(x) dx = j! \delta_{ij}. \]

(Proof: partial integration.)

The Hermite polynomials can be evaluated through the three-term recursion

\[ H_{j+1} = x H_j - j H_{j-1}. \]
Proof: induction

\[ H_0(x) = 1 \]
\[ H_1(x) = x \]
\[ H_2(x) = x^2 - 1 \]
\[ H_3(x) = x^3 - 2x \]

This implies the following form for the probability distribution function

\[ F(x) = N(x) + \sum_{k=1}^{\infty} c_k (-1)^k \phi^{k-1}(x) \]

and the moment generating function

\[ M(t) = e^{\frac{1}{2}t^2} \sum_{k=0}^{\infty} c_k t^k. \]

The Gram-Charlier coefficients can be expressed in terms of the moments:

\[ c_k = \sum_{j=0}^{[k/2]} (-1)^j \frac{\mu_{k-2j}}{(k-2j)!j!2^j}. \]

Closely related to the Gram-Charlier series is the Edgeworth expansion. Consider a sequence of i.i.d. random variables \( X_i \) with cumulants \( \kappa_r \). Then the cumulant-generating function of \( S = \frac{1}{\sqrt{n}}(X_1 + \)
\( \ldots + X_n - n \kappa_1 \) is

\[
nK(t/\sqrt{n}) = \frac{\kappa_2}{2} t^2 + \frac{\kappa_3}{3!} n^{-1/2} t^3 + \frac{\kappa_4}{4!} n^{-1} t^4 \ldots
\]

Going to the moment generating function and collecting terms w.r.t. powers of \( \sqrt{n} \) instead of powers of \( t \) gives the Edgeworth expansion:

\[
M(t/\sqrt{n})^n = e^{\frac{1}{2} \tilde{\kappa}_2 t^2} \sum_{k=0}^{\infty} n^{-k/2} P_k(t)
\]

where \( P_k \) are the Cramér-Edgeworth polynomials.

**Cornish-Fisher Expansion**

1. Normalize the random variable: \( \tilde{P} = (P - \kappa_1)/\sqrt{\kappa_2} \); i.e. \( \tilde{\kappa}_1 = 0, \tilde{\kappa}_r = \kappa_r \kappa_2^{-r/2} \)

2. Ansatz:

\[
\tilde{q}_\alpha = \sum_{k=0}^{\infty} \frac{d_k}{k!} z_\alpha^k.
\]

This implies

\[
N(z_\alpha) = \alpha = F(\sum_{k=1}^{\infty} \frac{d_k}{k!} z_\alpha^k).
\]
Idea: develop $\tilde{F}$ around $z_\alpha$ and invert.

Up to the fourth cumulant:

$$\tilde{q}_\alpha \approx -\frac{\tilde{\kappa}_3}{6} + \left(1 - \frac{\tilde{\kappa}_4}{8} + \frac{5\tilde{\kappa}_3}{36}\right)z_\alpha + \frac{\tilde{\kappa}_3}{6}z^2_\alpha + \left(\frac{\tilde{\kappa}_4}{24} - \frac{\tilde{\kappa}_3^2}{18}\right)z^3_\alpha$$

**Saddle Point Approximations**

several ideas combined:

1. exponential tilting
2. Laplace-approximation
3. path-integrals
4. uniform approximations

**Quadratic Programming**


$\overline{\text{VaR}}_\alpha :=$ worst case in a specific $\alpha$-confidence region (defined by iso-density lines)

- simply not the same as VaR
upper bound for the real VaR

The set \( \{ x \mid \Delta^\top x + \frac{1}{2} X^\top \Gamma X \} \) is quadratically bounded, but not necessarily convex.

**Monte-Carlo, Quasi-Monte-Carlo**

probably only competitive where Fourier inversion methods are infeasible because of large number of risk factors

**Conclusion**

1. Fourier inversion
   + easy to generalize to other distributions
   + fast in terms of accuracy
   – high “startup” cost (eigenvalue problem)

2. moment-based methods
   + lower startup costs
     i) fixed number of parameters
        i. Johnson distributions
        ii. Salomon-Stephens-approximation
   – somewhat ad-hoc
- fitting may not be robust

ii) series expansions
  i. Gram-Charlier series (Edgeworth series is related)
  ii. Cornish-Fisher expansion

3. Saddle-point methods
   + can be combined with both Fourier inversion and series expansion techniques
   ? supposed to drastically speed up convergence
random variable $X$ with distribution function $F$ and density $f$

**characteristic function:**

$$\phi_X(t) = Ee^{itX} = \int_{-\infty}^{\infty} e^{itx} F(dx) = \int_{-\infty}^{\infty} e^{itx} f(x)dx$$

- Fourier transform of the density $f$
- exists for $t \in \mathbb{R}$
- for complex arguments: If $\phi(z)$ exists for some $z \in \mathbb{C}$ then also for $z + a$, for all $a \in \mathbb{R}$.
- If $X \geq x_0$ then $\phi$ is defined on half-spaces $\{z | \text{Im}(z) \geq c\}$.

**Laplace transform:**

$$L_X(t) = Ee^{-tX}$$

(often only applied to nonnegative $X$)

**moment generating function:**

$$M_X(t) = Ee^{tX}$$
If $M_X(t)$ exists on an interval $t \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$, then

1. every moment exists
2. $M_X$ is analytic:

$$M_X(t) = \sum_{k=0}^{\infty} \frac{\mu_k}{k!} t^k$$

$$\mu_k = \mathbb{E}X^k = \frac{d}{dt} M_X(0).$$

cumulant generating function:

$$K_X(t) = \log \mathbb{E}e^{tX}$$

$M_X(t) > 0$ for $t \in \mathbb{R}$, so $K_X$ is analytic if $M_X$ is analytic. The power series coefficients are called cumulants:

$$K_X(t) = \sum_{r=1}^{\infty} \kappa_r \frac{t^r}{r!}$$

simple conversions:

$$\phi(-it) = L(-t) = M(t) = e^{K(t)}$$
Cumulants and moments can be expressed in terms of each other:

\[ \kappa_1 = \mu_1 \]

\[ \kappa_2 = \mu_2 - \mu_1^2 \]

\[ \kappa_3 = \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3 \]

\[ \kappa_4 = \ldots \]

\[ \kappa_k = \ldots \]

\( \frac{\mu_k}{k!} = \sum_{j=1}^{k} \frac{1}{j!} \sum_{r \in S(j,k)} \prod_{i=1}^{j} \frac{\kappa_{r_i}}{r_i!} \)

\( S(j,k) = \{(r_1, \ldots, r_j) | \sum_{i=1}^{j} r_i = k, r_i \in \{1, \ldots, k\}\} \)

**Multivariate version:**

\[ \phi_X(t) = E e^{i\langle t, x \rangle} \]

**Example**

\( X \) Gaussian \( N(\mu, \sigma) \):

\[ \phi_X(t) = e^{i\mu t - \sigma^2 t^2/2} \]

\[ K_X(t) = \mu t + \sigma^2 t^2/2 \]
multivariate:

\[ K_X(t) = \langle \mu, t \rangle + \frac{1}{2} \langle t, \Sigma t \rangle \]

Properties

1. \( \phi(0) = 1, \ |\phi(t)| \leq 1. \)
2. \( \phi(-t) = \overline{\phi(t)} \) for \( t \in \mathbb{R} \)
3. symmetric distribution implies that \( \phi(t) \) is symmetric and real (on \( \mathbb{R} \))
4. shifting: \( \phi_{X+c}(t) = E e^{it(X+c)} = e^{ict} \phi_X(t) \)
5. scaling: \( \phi_{aX}(t) = E e^{itaX} = \phi_X(at) \)
6. convolution: \( X, Y \) independent

\[ \phi_{X+Y}(t) = E e^{it(X+Y)} = E e^{itX} e^{itY} = \phi_X(t) \phi_Y(t) \]

\[ (f * g)(x) = \int_{-\infty}^{\infty} f(z) g(x-z) dz \]

\[ K_{X+Y}(t) = K_X(t) + K_Y(t) \]
7. inversion (and uniqueness (Levy 1925))

\[
f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} \phi_X(t) dt
\]

\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixt} M_X(it) dt
\]

\[
= \frac{1}{2\pi} \int_{-i\infty}^{+i\infty} e^{-xt} M_X(t) dt
\]

\[
F_X(b) - F_X(a) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-itb} - e^{-ita}}{-it} \phi_X(t) dt
\]

\( (F_X \) is the version with \( F_X(x) = (F_X(x+) + F_X(x-))/2 ) \.)

8. differentiation

\[
(\Phi f)'(t) = \int_{-\infty}^{\infty} e^{itx} ix f(x) dx = i(\Phi x f(x))(t)
\]

9. Riemann-Lebesgue-Lemma:

If \( X \) has a density, then \( \lim_{|t|\to\infty} \phi_X(t) = 0 \).

**Discrete Fourier Transform**

\[
(DFT h)(n) := H_n = \sum_{k=0}^{N-1} h_k e^{2\pi i kn/N}
\]
$H_n$ is periodic with period $N$

**inversion:**

$$h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{-2\pi i kn/N}$$

distribution on a regular grid:

$$P\{X = x_0 + \Delta k/N\} = h_k$$

$$\phi_X(t) = \mathbb{E}e^{itX} = \sum_{k=0}^{N-1} h_k e^{it(x_0 + \Delta k/N)}$$

$$= e^{ix_0 t} (\text{DFT}_h)(\Delta t/2\pi)$$
**Fast Fourier Transform**

first glance: matrix-vector multiplication \( \rightarrow O(N^2) \) operations

\[
H_n = \sum_{k=0}^{N-1} h_k e^{2\pi i kn/N}
\]

\[
= \sum_{k=0}^{N/2-1} h_{2k} e^{2\pi i 2k n/N} + \sum_{k=0}^{N/2-1} h_{2k+1} e^{2\pi i (2k+1) n/N}
\]

\[
= (\text{DFT} h^{\text{even}})(n) + e^{2\pi i n/N} (\text{DFT} h^{\text{odd}})(n)
\]

\( \rightarrow O(N \log_2 N) \) operations

**implementation:**

- implemented in most interactive packages like *S-Plus*…
- many tricks possible
- use the “Fastest Fourier Transform in the West” as a portable library
- Pentium II 300MHz, 65MFlops, 5 \( N \log_2 N \) operations:

<table>
<thead>
<tr>
<th>( N )</th>
<th>time in seconds</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2^{16} = 65536 )</td>
<td>0.08</td>
</tr>
<tr>
<td>( 2^{20} = 1048580 )</td>
<td>1.60</td>
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</tbody>
</table>
Further Reading

For properties on characteristic functions consult any text book on probability and statistics. For the fast Fourier transform see for example (Press et al.; 1992, p.500). For implementation issues consult (Frigo and Johnson; 1998).
References


Bundesaufsichtsamt für Kreditwesen (1997a). Bekanntmachung über die Änderung und Ergänzung der Grundsätze über das Eigenkapital und die Liquidität der Kreditinstitute vom 29. Oktober 1997 (Grundsatz I), was available as [http://www.bakred.de/texte/bekannt/grds1.htm](http://www.bakred.de/texte/bekannt/grds1.htm), now superseded by [http://www.bakred.de/texte/bekannt/g1_20070.htm](http://www.bakred.de/texte/bekannt/g1_20070.htm).


