Risk Management for Financial Institutions

Stefan R. Jaschke

Weierstraß-Institut für Angewandte Analysis und Stochastik

October 2000

Version: January 21, 2001

Introduction

The Practical Problem: Regulatory Capital

- choice between "standard method" and "internal model method" since 1998

Value at Risk = estimated quantile of the distribution of the portfolio's loss (over the next 10 days at the 99%-level)

$$\text{capital}_t \geq \max(f \cdot \frac{1}{60} \sum_{i=1}^{60} r_{t-i}, r_{t-1})$$

$f = 3 + f_1 + f_2$: "hysteria factor"
$0 \leq f_1, f_2 \leq 1$: additional loadings for qualitative and quantitative deficiencies
r_t: Value at Risk reported at t
(BAKred "Grundsatz I" 2000)

incentives:

1. lower required capital allows more business / higher expected returns
2. too low estimates of VaR can increase the factor f_1 (back-testing)

input: portfolio positions, contract specifications, and historical data for some 100000 securities

- commodities (precious metals, oil, ...)
- stocks
- fixed income, FX (depends on maturity and counter-party)
- futures (depends on maturity)
- options (depends on strike, maturity, underlying, ...)

aspects:

1. IT infrastructure (CORBA, database technology)
2. statistical modeling (dimension reduction, stochastic volatility)
3. computation (integration in high dimensions, approximation of distributions)

Why Important?

internal models:

1. lower capital requirements
2. approval by the BAKred can improve rating, standing, or volume
3. no need for separate internal and external control systems; the alternative (“standard model”) is too inflexible for derivatives

other uses of risk measures:

1. information reporting (shareholders, senior management)
2. performance measurement, compensation
3. resource allocation, controlling

for mathematicians:

- bigger market than the valuation and hedging of derivatives (anecdotal evidence)

Skeleton

1. The Need for Risk Management - Practical Problems
2. VaR and Coherent Risk Measures
3. VaR and Expected Shortfall for Portfolios
   i) independent multivariate normal risk factors (RiskMetrics)
   ii) independent identically distributed (iid) risk factor changes (historical simulation)
   iii) stochastic volatility models
   iv) (linear) dimension reduction techniques
   v) separate modeling of marginal distributions and dependence structure: correlations and copulas

4. Back-Testing
5. Stress-Testing

Where is the Math?

A. information based complexity (Traub and Werschulz; 1998) and applications to integration in high dimensions
B. extreme value theory (Embrechts et al.; 1997)

IBC Example

How many function evaluations are needed to achieve a certain level of accuracy ϵ?

! multivariate integration is intractable in the worst-case setting:

$$\text{complexity}^{worst}(\epsilon, d) \sim c(f, d)\epsilon^{-d/r}$$

(Bakhvalov 1959)
$c(f, d)$: cost of function evaluation
ϵ: approximation accuracy
d: dimension
r: derivatives up to degree r are bounded

- similar in spirit to complexity theory based on Turing-machine: provides lower bounds; classifies problems

- very important alternative complexity theory

EVT Example

$X_1, \ldots, X_n$ iid $\sim F$

$$\frac{\max\{X_1, \ldots, X_n\} - a_n}{b_n} \xrightarrow{d} H$$

- only *extreme value distributions* (Gumbel, Weibull, Frechet) can appear as the limit H
- virtually all known distributions are in the *maximum domain of attraction* of one of the extreme value distributions

(Fisher-Tippett 1928)

- long-standing applications in hydrology

Part II

6. Representations of Partial Preferences
7. Valuation Bounds in Incomplete Markets

Portfolio Optimization

standard: Markowitz, CAPM, based on μ, σ

alternative: (μ, ρ)-optimization for general risk measures ρ:

$$\max_{x - x_0 \in M} \{\mu(x) \,|\, \rho(x) \leq c\}$$

(Markowitz; 1959)

Valuation and Hedging of Derivatives

standard: Black/Scholes, complete market models

alternative: hedging is special case of PO-problem
$\rightarrow$ optimal hedge & price bounds depend on: initial position, transaction costs, trading constraints, . . .
(Hodges and Neuberger; 1989)

! important to first know how to *measure* risk before formulating optimization problems based on risk measures

guess: VaR-infrastructure/IT-investments allow to actually implement some of the incomplete market models

The Need for Risk Management

The Historic View

sudden increase in financial market risks

1971 breakdown of the fixed exchange rate system

1973 oil price shock, volatile interest rates

financial derivatives

1972 FX futures traded in the "International Monetary Market" (IMM) at the Chicago Mercantile Exchange (CME)

1973 stock options traded at the Chicago Board Options Exchange (CBOE), founded by members of the Chicago Board of Trade (CBOT)

1975 GNMA futures (CBOT)

1977 Treasury bond futures (CBOT)

1982 options on T-bond futures (CBOT)

1983 stock index options (CBOE)

(Further events see (Jorion; 2000, Table 1-2, p.13).)

option pricing theory

1973 Black and Scholes (geometric Brownian motion, based on PDE arguments)

1976 Cox and Ross (alternative processes, probabilistic arguments)

1979 Harrison and Kreps ("nice" martingale-theoretic understanding)

1987 stock market crash
breakdown of portfolio insurance
pronounced volatility smile ever since

regulatory action

1974 collapse of the Herstatt bank,"Basel Committee on Banking Regulations and Supervisory Practices" founded by central-bank Governors of the G-10 countries

Jul 1988 Basel Accord (BCBS88)
goal: secure bank deposits, control of credit risks
"capital" $\geq$ 8% of "risk-weighted assets"
restriction of "large risk"
$\rightarrow$ static, diversification ignored

1989 EU "Solvency Ratio" and "Own Funds" Directives

1991 Federal Deposit Insurance Corporation Improvement Act

Apr 1993 (BCBS93): first proposal of the "building-block approach", now called "standard method", superseded by (BCBS95)

- goal: control of market risks
- decomposition: commodity risk, FX rate risk, interest rate risk, equity risk
- "capital" $\geq$ 8% of "risk-weighted assets"

$\rightarrow$ still static, some diversification

Mar 1993 Capital Adequacy Directive (EU Council)

the disaster period

Feb 1993 Showa Shell \$1580 in currency forwards

Jan 1994 Metallgesellschaft \$1340 in oil futures

Apr 1994 Kashima Oil \$1450 in currency forwards

Dec 1994 Orange County \$1810 in reverse repos

Feb 1995 Barings \$1330 in stock index futures (further losses and stories in (Jorion; 2000, chapter 2))

flurry of private sector and regulatory activity

1993 G-30 report "Derivatives: Practices and Principles"

May 1994 General Accounting Office report

1994 Derivatives Policies Group: "Framework for Voluntary Oversight"

Oct 1994 JP Morgan: RiskMetrics

regulatory action

Apr 1995 (BCBS95): proposal for *internal models*

Jan 1996 (BCBS96): Amendment to the Basel Accord (internal models) (effective January 1998)

Oct 1997 BAKred: "Grundsatz I" (Bundesaufsichtsamt für Kreditwesen; 1997a) (effective October 1998)

Jun 1998 CAD 2 (EU Council; 1998a)

Jul 2000 BAKred: "Grundsatz I" (Bundesaufsichtsamt für Kreditwesen; 2000)

Current Rules

Grundsatz I, §33:

internal model covers "general market risks" only or includes also "specific event risk":

$$\mathrm{MRC}_t = \max(f \cdot \frac{1}{60}\sum_{i=1}^{60} r_{t-i}, r_{t-1})$$

internal models covers "specific market risks", but not "event risk":

$$\mathrm{MRC}_t = \max(f \cdot \frac{1}{60}\sum_{i=1}^{60} r_{t-i} + \frac{1}{60}\sum_{i=1}^{60} s_{t-i}, r_{t-1} + s_{t-1})$$

(s_t is the "specific risk" surcharge at t.)

- written approval by BAKred (§32)
- 10 days horizon, 99% probability level, based on at least 1 year's historic data (§34)
- "sufficient" set of risk factors: risks associated with nonlinear instruments, term structure of interest rates and spread risks, spot-forward spreads for commodities (§35)
- qualitative requirements (§36)
- backtesting (§37): daily VaR is compared to actual exceedences in the last 250 days ("clean P&L") add-on factor f_1:

traffic light	exceedences	factor
green	≤ 5	0.00
yellow	5	0.40
	6	0.50
	7	0.65
	8	0.75
	9	0.85
red	≥ 10	1.00

Outlook

consequences for

- portfolio optimization
- hedging and valuation of derivatives
- incomplete market models

not yet implemented in law

1995 Fed: pre-commitment approach (Kupiec and O'Brien)
no need for "clean P&L" computations

Jun 1999 (BCBS99b) "A New Capital Adequacy Framework"
"three pillars":

1. minimum capital requirements (external credit assessments)
2. a supervisory review process
3. effective use of market discipline

"maybe": internal credit ratings and portfolio credit risk modeling

Nov 1999 European Commission (1999): "A Review ..."

Jan 2000 (BCBS2000a): pillar 3, more disclosure

Sep 2000 (BCBS2000c): credit risk modeling

Suggested Homework

Skim the first 6 sections of the "Grundsatz I" (Bundesaufsichtsamt für Kreditwesen; 2000) to get an overview of the "standard method" (and appreciate the alternative "internal model method"). Read section 7 containing the current implementation of the "internal model method" in German law.

Further Reading

A good overview of the developments leading to regulatory Value at Risk is given by (Jorion; 2000, chapters 1-3). Chicago was the birth place for exchange traded derivatives, see the web sites Chicago Board of Trade (2000), Chicago Mercantile Exchange (2000a), and Chicago Board Options Exchange (2000). Davies has a nice collection of web links to documents about financial scandals. The texts of the Basel Committee on Banking Supervision are available from http://www.bis.org/publ/pub_list.htm. (Basel Committee on Banking Supervision; 2000a) gives a short historic view of the Committee's activities and main proposals. The EU consultation documents are available from http://europa.eu.int/comm/internal_market/en/finances/banks/, directives from http://europa.eu.int/eur-lex/. Most German legal texts regarding banking regulation are available from http://www.bakred.de. Traber (2000) provides insight into the process of the approval of internal models from the viewpoint of a regulator.

VaR and Other Risk Measures

VaR

usage of the term *Value at Risk*:

1. VaR is an estimate of some upper bound on the likely loss of a portfolio's market value over a target horizon.

$$= \rho(-\mathrm{P\&L}(\phi))$$

2. ► VaR is the estimated quantile of the portfolio's loss over a target horizon.

$$\mathrm{VaR}_\alpha(\phi) = q_\alpha(-\mathrm{P\&L}(\phi))$$

   i) ► q_α is an α-*quantile* of the distribution of a random variable X (under the probability P) if

$$P\{X < q_\alpha\} \leq \alpha \leq P\{X \leq q_\alpha\}.$$

   ! Quantiles for a fixed alpha form a closed interval.

   ii) "VaR summarizes the expected maximum loss over a target horizon within a given confidence interval." (Jorion; 1997, p.19)

$$q^-_\alpha(X) = \inf\{x \,|\, P\{X \leq x\} \geq \alpha\}$$

3. VaR is the methodology of assuming normally distributed driving factors and expressing the portfolio value as a function of these underlying factors (as in RiskMetrics). ("...since VaR accounts for correlations ..." (Jorion; 1997, p.285). "The other characteristic of VaR is that it takes account of correlations ..." (Dowd; 1998, p.20).)

confidence level:

- 99% (G I)
- 95% (RiskMetrics)

horizon:

- 1 day (G I: for backtesting; RiskMetrics: for trading)
- 10 days (G I: for required capital)
- 25 days (RiskMetrics: for investment)

advantages:

- simple
- single number, can aggregate different kinds of risk, allows enterprise-wide risk management (ERM)
- in \$, translation property (unlike σ)

$$\rho(X + \alpha\mathbf{1}) = \rho(X) - \alpha, \quad \forall \alpha \geq 0.$$

VaR Is the Wrong Risk Measure ...

... for trading limits and the allocation of resources

(1) VaR is not compatible with diversification.

- defaultable bonds (Artzner et al.; 1998, p.14, attributed to Albanese)

... for compensation

(2) VaR induces the wrong incentives in reasonably complete markets.

- buy defaultable bonds, sell riskless bonds (LTCM)
- sell far-out-of-the-money puts
- doubling strategy
- delta-hedged OTC book: short gamma, short vega
- Peso problem traders (Taleb; 2000)

... for regulatory purposes

(3) VaR is the *minimal* loss of the $(100 - \alpha)\%$ "bad" cases. It says nothing about the *average* loss in the "bad" cases.

- OK from the viewpoint of management: interested in a long life of the firm. (Similar for shareholders.)

- Not OK from the viewpoint of the central bank and the taxpayer: how much does the cleanup cost (on average)?

alternative: *tail conditional expectation* (alias "tail VaR", "conditional VaR", or "beyond VaR"):

$$\mathrm{TCE}_\alpha(\phi) = \mathrm{E}[-\mathrm{P\&L}(\phi) \,|\, -\mathrm{P\&L}(\phi) \geq \mathrm{VaR}_\alpha(\phi)],$$

which is related to *expected shortfall* (alias "mean excess loss")

$$\mathrm{ES}_\alpha(\phi) = \mathrm{E}[-\mathrm{P\&L}(\phi) - \mathrm{VaR}_\alpha(\phi) \,|\, -\mathrm{P\&L}(\phi) \geq \mathrm{VaR}_\alpha(\phi)]$$

by

$$\mathrm{TCE}_\alpha = \mathrm{VaR}_\alpha + \mathrm{ES}_\alpha.$$

Life Expectancy as a Function of the VaR Level

life time:

$$\tau = \min\{t \,|\, c + \sum_{i=1}^{t} X_i \leq 0\}$$

X_i	iid
c	initial capital

→ "ruin theory", "risk theory" (Cramer, Lundberg)

• not stationary, not realistic

alternative: "take more business as we grow, stay at the VaR-limit":

$$\tau = \min\{t \,|\, c + X_t \leq 0\}$$

($P\{-X_t \geq c\} = \alpha$ is the VaR confidence level.)
τ has a geometric distribution: $P\{\tau = n\} = \alpha^{n-1}(1-\alpha)$
$\rightarrow \mathrm{E}[\tau] = \frac{1}{1-\alpha}$

99% VaR, 10 days horizon, life expectancy in years:

factor m	normal	exponential	power
1	4	4	4
3	$2.7 \cdot 10^{10}$	$4 \cdot 10^4$	36
4	$6.0 \cdot 10^{18}$	$4 \cdot 10^6$	64
5	$2.8 \cdot 10^{29}$	$4 \cdot 10^8$	100

(Tails are (1) standard normal, (2) exponential ($e^x \mathbf{1}_{[-\infty,0]}$), (3) power ($2|x|^{-3}\mathbf{1}_{[-\infty,-1]}$).)

Conclusion

VaR is blind towards large losses with small probabilities. This is appropriate in situations were only a small ruin probability – alias a long life expectancy – is desired, i.e. the costs associated with the ruin are essentially fixed and do not depend on the size of the exceedence. This is the case for the overflowing of dikes. It is also approximately the case for bankruptcies from the viewpoint of management (=costs of looking for a new job) as well as the viewpoint of shareholders (=liquidation costs). It is markedly not the case for bankruptcies of banks from the viewpoint of depositors, creditors, regulatory authorities, tax payers, or whoever has to bear the loss that exceeds the capital of the failed bank.

Computation of VaR and ES for specific distributions

$$\mathrm{VaR}_\alpha(X) = -\sup\{q \,|\, \mathrm{E}[\mathbf{1}_{\{X \le q\}}] \le 1-\alpha\}$$

"largest strike price such that the price of a digitial put option is at most $1-\alpha$"

$$\mathrm{ES}_\alpha(X) = \frac{1}{1-\alpha}\mathrm{E}[(X - \mathrm{VaR}_\alpha(X))^-]$$

"price of a put option with strike at the VaR-level divided by the price of the corresponding digital option"

(In a risk-neutral world.)

! Option valuation techniques can be applied to risk measurement and vice versa. Where risk measurement techniques are especially suited to measure tail behavior, they are also especially suited to value out-of-the-money options.

1 Standard Normal Distribution

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$\begin{aligned}
\mathrm{VaR}_\alpha &= q_\alpha \\
\mathrm{TCE}_\alpha &= \frac{1}{1-\alpha} \int_{-\infty}^{q_{1-\alpha}} -x\phi(x)dx \\
&= \frac{\phi(q_{1-\alpha})}{1-\alpha}
\end{aligned}$$

($\phi'(x) = -x\phi(x)$.)

2 Exponential Tails

$$\begin{aligned}
f(x) &= \lambda e^{\lambda x} \mathbf{1}_{(-\infty,0)} \\
F(x) &= \max(e^{\lambda x}, 1)
\end{aligned}$$

($\lambda > 0$.)

$$\begin{aligned}
\mathrm{VaR}_\alpha &= -\frac{1}{\lambda} \log(1-\alpha) \\
\mathrm{TCE}_\alpha &= \frac{1}{\lambda} + \mathrm{VaR}_\alpha
\end{aligned}$$

$\rightarrow$ constant expected shortfall = absolute difference between VaR and TCE

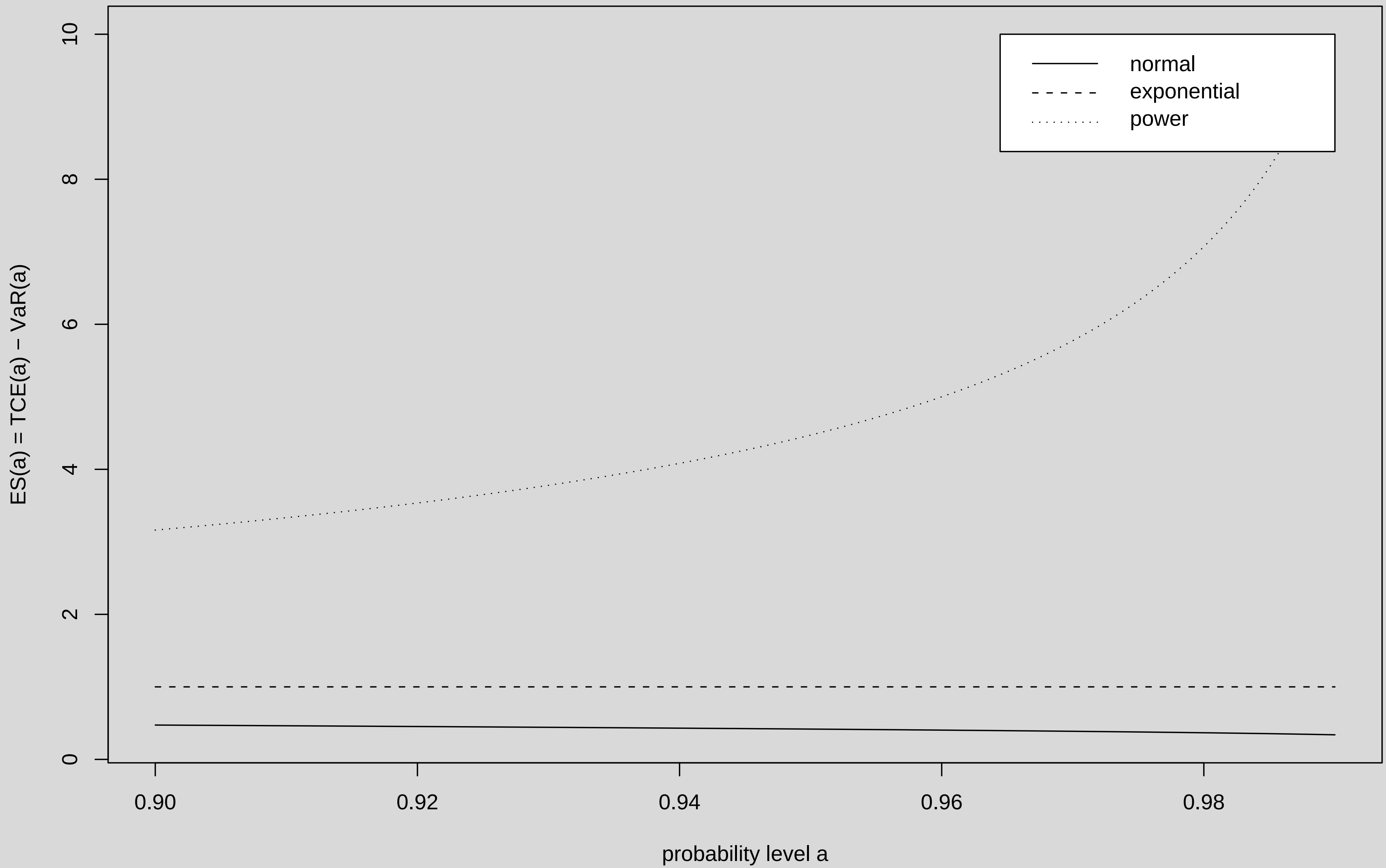
3 Power Tails

$$
\begin{aligned}
f(x) &= \beta(-x)^{-\beta-1}\mathbf{1}_{(-\infty,-1)} \\
F(x) &= \max((-x)^{-\beta}, 1)
\end{aligned}
$$

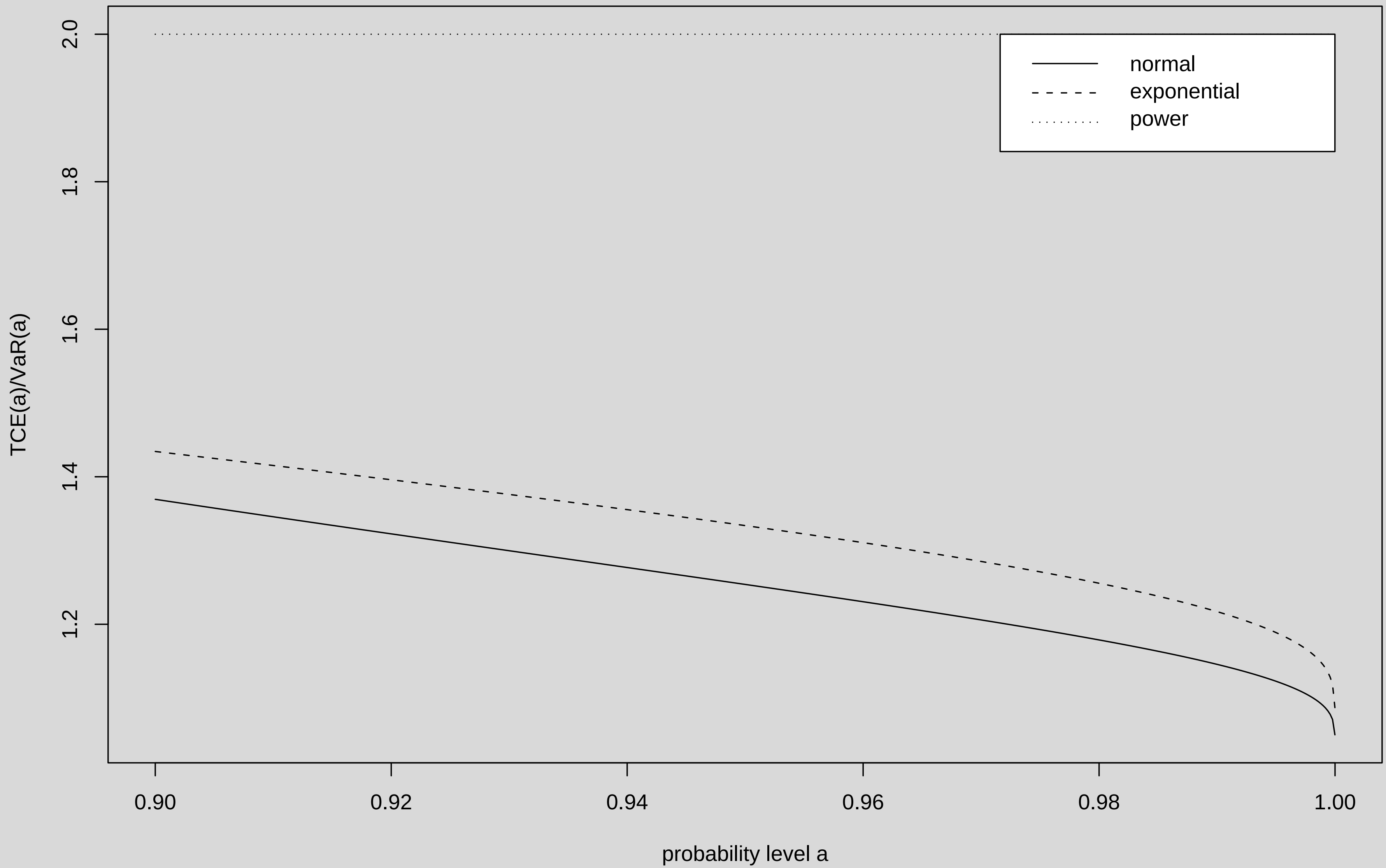
$(\beta > 1.)$

$$
\begin{aligned}
\mathrm{VaR}_\alpha &= (1-\alpha)^{-1/\beta} \\
\mathrm{TCE}_\alpha &= \frac{\beta}{\beta-1}\mathrm{VaR}_\alpha
\end{aligned}
$$

$\rightarrow$ constant relative difference between VaR and TCE



ES(a) = TCE(a) − VaR(a)
0
2
4
6
8
10
0.90
0.92
0.94
0.96
0.98
probability level a
normal
exponential
power



normal
exponential
power
2.0
1.8
1.6
1.4
1.2
TCE(a)/VaR(a)
0.90
0.92
0.94
0.96
0.98
1.00
probability level a

But: Any level of TCE can be reached by “Peso problem strategies” with restricted VaR:

$$\mathrm{P\&L} = \begin{cases} 1 & \text{with probability } 0.5 \\ -1 & 0.5 - p \\ -x & p \end{cases}$$

with $p < 1 - \alpha < 0.5$ and $x > 1$. Then the VaR is constant

$$\mathrm{VaR}_\alpha = 1$$

but the TCE is unbounded:

$$\mathrm{TCE}_\alpha = 1 + 2(x - 1)p.$$

Coherent Risk Measures

Artzner et al. (1998): What are desirable properties of risk measures (for regulatory purposes)?

Ω finite. Space of random variables: $L = \{X | X : \Omega \to \mathbb{R}\}$. Risk-less return: $\mathbf{1}(\omega) \equiv 1 + r$.

- A *risk measure* $\rho : L \to \mathbb{R}$ is called *coherent* if it has the following four properties:

1. translation invariance

$$\rho(x + \alpha\mathbf{1}) = \rho(x) - \alpha$$

$+$ VaR

$-$ not: σ, entropy

2. convexity

$$\rho(\lambda x + (1-\lambda)y) \leq \lambda\rho(x) + (1-\lambda)\rho(y)$$

$=$ consistent with diversification and risk aversion

3. positive homogeneity

$$\rho(\alpha x) = \alpha\rho(x),\ \alpha > 0$$

$-$ contradicts usual way of thinking about risk aversion

$+$ OK from viewpoint of regulators

4. monotonicity

$$x \in L^+ \implies \rho(x) \leq 0$$

$(\alpha \in \mathbb{R}; x, y \in L; \lambda \in [0,1].)$

One-to-one Correspondences

informally:

! Modulo technical conditions, there is a one-to-one correspondence between the following economic objects:

1. "coherent risk measures" ρ,
2. cones of "acceptable risks" ($A = \{x \,|\, \rho(x) \leq 0\}$),
3. partial orderings "$x \succeq y$", meaning "x is at least as good as y" ($x \succeq y \iff \rho(x-y) \leq 0$),
4. valuation bounds $\overline{\pi}$ and $\underline{\pi}$ (with $\rho(x) = \overline{\pi}(-x) = -\underline{\pi}(x)$), and
5. sets K of "admissible" price systems π ($\pi \in K \iff \pi(x) \geq 0$ for all $x \succeq 0$).

- *coherent acceptance set*:

(a) A is a cone,
(C) A is closed and
(M) $L^+ \subset A$.

- *coherent partial preferences*:

(a) $\succeq$ is a vector ordering
(C) $\{x \,|\, x \succeq y\}$ is closed, and
(M) $x \geq 0 \implies x \succeq 0$

- *coherent valuation bounds:*

$\overline{\pi}(x) = -\underline{\pi}(-x)$ and $-\underline{\pi}$ is a coherent risk measure.

▶ *coherent set of admissible price systems*:

$$\begin{array}{ll} (a) & K \text{ is a cone,} \\ (C) & K \text{ is closed, and} \\ (M) & x \in L^{+} \implies \pi(x) \geq 0 \text{ for all } \pi \in K. \end{array}$$

formally:

! There is a one-to-one correspondence between coherent risk measures, coherent acceptance sets, coherent partial preferences, coherent valuation bounds, and coherent sets of admissible price systems.

▶ A *normalized price system* π is a linear mapping from L to $\mathbb{R}$ with $\pi(\mathbf{1}) = 1$ and $x \geq 0 \implies \pi(x) \geq 0$.

Representation Theorem

! There is a one-to-one correspondence between normalized, non-negative price systems and probability measures Q by the mappings

$$\begin{aligned} \pi(X) &= \mathrm{E}_P[x/(1+r)] \quad \text{and} \\ P(A) &= \pi(\mathbf{1}_A), \end{aligned}$$

where $\mathbf{1}_A = (1+r)\chi_A$ and χ_A is the indicator function of A.

! strong duality theorem:

Let A be a closed cone containing $\mathbf{1}$, $K = A^{\top}$ its polar cone, and

$$(P) \qquad \rho(x) = \inf\{p \,|\, p\mathbf{1} + x \in A\}$$

the associated risk measure.

(i) The set of normalized admissible price systems

$$D := \{\pi \in K \,|\, \pi(\mathbf{1}) = 1\}$$

is not empty iff $-\mathbf{1} \notin A$ ($\rho > -\infty$).

(ii) In this case

$$(D) \qquad \rho(X) = \sup_{\pi \in D} -\pi(X).$$

Analogously:

$$\begin{aligned} \overline{\pi}(X) &= \sup_{\pi \in D} \pi(X) \\ \underline{\pi}(X) &= \inf_{\pi \in D} \pi(X) \end{aligned}$$

$\rightarrow$ "*extended present value principle*"

Using the one-to-one correspondence between normalized price systems and probability measures, the duality theorem implies the

! representation theorem:

ρ is a c.r.m. iff there exists a set $\mathcal{Q}$ of probability measures:

$$\rho(X) = \sup_{Q \in \mathcal{Q}} -\mathrm{E}_Q[X/(1+r)]$$

Examples

Chicago Mercantile Exchange (2000b):

- based on scenarios, of the form $\sup_{Q \in \mathcal{Q}} -\mathrm{E}_Q[X]$

The SEC rules:

- benchmark securities Y_i
- assign risk numbers ρ_i

▶ $A := \mathrm{cone}\{Y_i + \rho_i \mathbf{1}\} + L^+$

! A is coherent

worst conditional expectation:

$$\mathrm{WCE}_\alpha(X) := \sup_{B: P(B) \geq 1-\alpha} E_P[-X/(1+r) \mid B].$$

! TCE_α is a “solution” to this optimization problem (exact in special cases, approximation otherwise, see (Artzner et al.; 1998)).

General Risk Measures

- *partial preferences* $\succeq$:

1. $\succeq$ is a pre-order,
2. the level sets $\{y \mid y \succeq x\}$ are convex,
3. $\succeq$ is monotone: $x \geq y$ implies $x \succeq y$.

- associated *acceptance set* $A(e)$ (depending on the initial endowment e):

$$A(e) = \{x \mid x + e \succeq e\}.$$

- associated *risk measure* $\rho(.,e)$:

$$\rho(x,e) = \inf_{p \in \mathbb{R}} \{p \mid p\mathbf{1} + x + e \succeq e\}$$

VaR is the Industry Standard

- legal texts (G I)
- RiskMetrics is the benchmark
- widely implemented

! ES may be used as a complement to VaR in the future (Härdle and Stahl; 1999).

Further Reading

Artzner et al. (1998) introduce coherent risk measures and provide lots of economic arguments. Jaschke and Küchler (1999) generalize the concept to linear spaces and add the relation between valuation bounds and risk measures. (Jaschke and Küchler; 1999) also contains some examples of risk measures and partial orderings that are not coherent.

Nassim Taleb is well-known for his incisive critique of (the usual/naive application of) Value at Risk. Many important insights into risk management in general are contained in the first (online) sections of (Taleb; 2000). It contains some real life examples of traders who continuously earned several millions per year for their firms over many years and hence were very highly valued. Then they lost a multiple of what they had earned over the previous years in a few weeks. (Needless to say that they lost their jobs.) As a matter of fact, there are trading strategies that produce long stretches of low-volatility excess returns, but then crash big. Because buying emerging market bonds is one way to achieve such behavior Taleb calls traders running such strategies “Peso problem traders”. Now the point is to observe that Value at Risk (in both the narrower and wider senses) fails to “see” the risk in these “Peso problem strategies”.

VaR and TCE for Portfolios

The Usual Classification

Longerstaey (1996); Jorion (1997); Dowd (1998):

1. "Variance-Covariance Approach", "Delta- (Gamma-) Normal", "Analytic Method"
2. "Historical Simulation"
3. "Structured Monte Carlo"

Variance-Covariance Approach

modeling decision:

$$\mathrm{P\&L}_t(\phi, X) = \sum_i \phi_i f_i(X_t)$$

The innovations in the underlying risk factors X_t are iid multi-variate normal $N(\mu, \Sigma)$. μ is usually 0.

estimation decision: Estimate Σ as moving average or exponentially weighted moving average (EWMA) of empirical covariances.

approximation decision:

$$f_i(x) \approx f_i(0) + \sum_{j=1}^{m} x_j \frac{\partial f_i}{\partial x_j}(0) \qquad \text{(Delta-Normal)}$$

$$f_i(x) \approx f_i(0) + \sum_{j=1}^{m} x_j \frac{\partial f_i}{\partial x_j}(0) + \sum_{j=1}^{m}\sum_{k=1}^{m} x_j x_k \frac{\partial^2 f_i}{\partial x_j \partial x_k}(0)$$

(Delta-Gamma-Normal)

Computation of VaR and TCE (for Delta-Normal)

$(f_i(0) = 0, \mu = 0)$

$$\begin{aligned} \text{P\&L} &\approx \sum_i \phi_i \sum_{j=1}^{m} X_j \frac{\partial f_i}{\partial x_j}(0) \\ &= \Delta^\top X \end{aligned}$$

with $\Delta_j = \sum_i \phi_i \frac{\partial f_i}{\partial x_j}(0)$.
! $\Delta^\top X \sim N(0, \Delta^\top \Sigma \Delta)$

$\Longrightarrow$

$$\begin{aligned} \mathrm{VaR}_\alpha(\phi) &= q_\alpha(N(0,1)) \cdot \sigma \\ \mathrm{TCE}_\alpha(\phi) &= \mathrm{TCE}_\alpha(N(0,1)) \cdot \sigma \end{aligned}$$

with $\sigma = \sqrt{\Delta^\top \Sigma \Delta}$.

Historical Simulation

modeling decision:

$$\mathrm{P\&L}_t(\phi, X) = \sum_i \phi_i f_i(X_t)$$

The innovations in the underlying risk factors X_t are iid $\sim F$, not necessarily normal.

estimation decision: Take the empirical distribution function of past observations X_{t-k} as an estimator for F. Plug the historic samples into the $\mathrm{P\&L}$-equation and use empirical quantiles, smoothing techniques, or EVT to estimate the quantile of the $\mathrm{P\&L}$-distribution function.

no **approximation** necessary

Structured Monte Carlo

statistical and estimation decisions: Unspecified (but in practice often just iid multi-variate normal increments).

approximation decision: By using *pseudo random numbers* or *quasi random numbers* generate a sample $(x_1, \ldots, x_n)$ according to the assumed distribution of X_t. Plug the samples into the $\mathrm{P\&L}$-equation and use empirical quantiles to estimate the quantile of the $\mathrm{P\&L}$-distribution function.

Separation of Different Aspects

Critique:

- statistical and numerical decisions not well separated
- methods are usually compared on few selected portfolios $\longrightarrow$ anecdotal evidence only

statistical		numerical
which model?	which estimator?	how to approximate/ compute?
depends *essentially* on the statistical properties of market data		given a model, depends *only* on the portfolio or class of possible portfolios

$\longrightarrow$ Compare statistical decisions w.r.t. a model choice criterion by using exact computations on the computation side.

$\longrightarrow$ In a given model, compare approximation techniques w.r.t. worst-case or average case approximation errors over classes of portfolios.

Mapping

1. *mapping in the narrow sense*: how to choose f_i for given risk factors X (like in RiskMetrics) in

$$\mathrm{P\&L}(\phi) = \sum_i \phi_i f_i(X).$$

2. *mapping in the wide sense*: how to choose X in the first place
   = dimension reduction
   - statistical decision (factor analysis, subspace methods)

Mapping in the narrow sense (Longerstaey; 1996, chapter 6)

Step 1: break down contracts into individual payments

- bond = portfolio of zero-bonds
- FRN = zero-bond, maturing at the next settlement date
- swap = bond - FRN
- FRA(t_1, t_2)= zero-bond(t_1) - zero-bond(t_2)
- same for interest rate futures
- FX forward = zero-bond(t, c_1) - zero-bond(t, c_2)
- currency swaps = bond(c_1) - FRN(c_2)
- commodities: forwards and swaps like currency forwards and swaps

- options: $f_i =$ Black-Scholes-formula; risk factors: the underlying, the implied volatility, the interest rate

$\rightarrow$ mapping positions in contracts to positions in "building blocks":

- single shares
- single deterministic future payments
- single future payments/delivery in foreign currencies or commodities
- implied volatilities

Step 2: map positions in building blocks to positions in risk factors

1. equity, FX rates, commodity prices
   i) "building block" is a risk factor itself
      i. log returns are assumed to be normally distributed:

$$\begin{aligned}
\text{P\&L} &= P_t + D_t - P_{t-1} \\
r_t &= \log((P_t + D_t)/P_{t-1}) \\
f(x) &= e^x - 1 \\
\text{P\&L} &= P_{t-1} f(r_t)
\end{aligned}$$

ii. percent returns are assumed to be normally distributed:

$$\begin{aligned} R_t &= (P_t + D_t - P_{t-1})/P_{t-1} \\ f(x)) &= x \\ \mathrm{P\&L} &= P_{t-1} f(R_t) \end{aligned}$$

	percent returns	log returns
normality OK	$-$	$+$
f linear	$+$	$-$
FX cross rates OK	$-$	$+$

ii) mapping to an index (CAPM-like)

$$\begin{aligned} r_t &= \beta r_{m,t} + \mu + \epsilon_t \\ f(x) &= e^{\beta x + \mu} - 1 \\ \mathrm{P\&L} &\approx P_{t-1} f(r_{m,t}) \end{aligned}$$

(ignoring the firm-specific risk ϵ_t)

2. zero-bonds

i) exact maturity is a risk factor

i. log returns are normal

- RiskMetrics
- somewhat unrealistic since price ≤ 1

ii. changes in yields are log-normal:

$$\begin{aligned} P_t(\tau) &= (1 + y_t(\tau))^{-\tau} \\ x_t &= \log(y_{t+1}(\tau-1)/y_t(\tau)) \sim N(0,1) \\ \text{P\&L} &= P_{t+1}(\tau-1) - P_t(\tau) \\ &= (1 + e^x(P_t(\tau)^{-1/\tau} - 1))^{-\tau} - P_t(\tau) \\ &= f(x, P_t) \end{aligned}$$

ii) mapping to selected maturities: (1m 3m 6m 1y 2y 3y 4y 5y 7y 9y 10y 15y 20y 30y)

i. (Longerstaey; 1996):

A. value is preserved (yields are linearly interpolated)

B. volatility is preserved (volas are linearly interpolated)

C. sign is preserved

$\rightarrow$ unique solution of a quadratic equation

! problem: Map is discontinuous if correlation between vertices is low.

ii. Mina (1999): simple linear mapping:

$$d_\tau = \alpha d_{\tau_L} + (1-\alpha) d_{\tau_R}$$

with

$$\alpha = \frac{\tau_R - \tau}{\tau_R - \tau_L}.$$

A. does not have the discontinuity problem of the standard RiskMetrics mapping method

iii. nonlinear mapping, e.g. yield is linearly interpolated $\rightarrow$ interpolated discount factors are not a linear function of discount factors with selected maturities

3. later delivery of commodities, implied volatilities
    - roughly similar to zero-bond-mapping

Conclusion

The discussion in both Longerstaey (1996) and Mina (1999) is somewhat limited as it focuses on linear mapping.

Mapping (in the narrow sense) is highly dependent on the choice of risk factors (mapping in the wider sense) and should be viewed in the context of (statistical) dimension reduction techniques.

Delta-Gamma-Methods

risk factors $X \sim N(0, \Sigma)$

$$P = \theta^\top \mathbf{1} + \Delta^\top X + \frac{1}{2} X^\top \Gamma X$$

These "Greeks" (=derivatives w.r.t. the risk factors) have to be computed from the usual Greeks (=derivatives w.r.t. the underlyings) by use of the chain rule of differentiation.

Fourier Inversion

1.step: Express P as sum of independent non-central χ_1^2 variates.

Choose linear transformation $X = CY$ such that

$$(1) \qquad Y \sim N(0, I)$$

$$(2) \qquad P = \theta^\top \mathbf{1} + (C^\top \Delta)^\top Y + \frac{1}{2} Y^\top \Lambda Y$$

for a diagonal Λ. This is a *generalized symmetric eigenvalue problem*:

$$(1) \qquad CC^\top = \Sigma$$

$$(2) \qquad C^\top \Gamma C = \Lambda$$

(The columns of C contain the generalized eigenvectors, Λ contains the eigenvalues.)

Packages like Anderson et al. (1999) contain routines directly for the generalized eigenvalue problem. Otherwise C can be computed in two steps:

1. Compute a matrix B with $BB^\top = \Sigma$, preferably by Cholesky decomposition.
2. Solve the (standard) symmetric eigenvalue problem for the matrix $B^\top \Gamma B$:

   $$Q^\top B^\top \Gamma B Q = \Lambda$$

   and set $C := BQ$.

P is sum of independent random variables:

$$\begin{aligned} P &= \sum_i (\theta_i + \delta_i Y_i + \frac{1}{2}\lambda_i Y_i^2) \\ &= \sum_i \left\{ \theta_i + \frac{1}{2}\lambda_i \left(\frac{\delta_i}{\lambda_i} + Y_i \right)^2 - \frac{\delta_i^2}{2\lambda_i} \right\} \end{aligned}$$

with $\delta = C^\top \Delta$.

2.step: compute the mgf of P (= product of mgfs of non-central χ_1^2 variates)

The moment generating function of a non-central χ_1^2 is known analytically:

$$\mathrm{E} e^{t(Z+a)^2} = (1-2t)^{-1/2} \exp\left(\frac{a^2 t}{1-2t} \right) \qquad (t < \frac{1}{2})$$

($Z \sim N(0,1)$.) This implies the mgf for P:

$$\mathrm{E}e^{tP} = \prod_i \exp\left(\theta_i t + \frac{1}{2}\delta_i^2 t^2 (1-\lambda_i t)^{-1} - \frac{1}{2}\log(1-\lambda_i t)\right).$$

Re-expressed in terms of Γ and Σ:

$$\log \mathrm{E}e^{tP} = \theta^\top \mathbf{1} t + \frac{1}{2}t^2 \Delta^\top \Sigma (I - \Gamma\Sigma t)^{-1}\Delta - \frac{1}{2}\mathrm{tr}(\log(I - \Gamma\Sigma t)). \quad (1)$$

(Mina and Ulmer (1999) seem to have some errors there.)

3.step: Fourier inversion

operation count (m risk factors, N discretization points):

generalized eigenvalue problem:	$O(m^3)$
mgf:	$O(mN)$
FFT:	$O(N \log N)$

eigenvalue problem dominates: 3 minutes on a Pentium II 450 MHz for 500 risk factors

Summary

- This technique is more general then just Gaussian: works well if characteristic function is known.
- It works even if the characteristic function is not known but the distributions of the Y_i's can be

estimated and the Y_i's can be assumed independent. Gradients (sensitivities) of VaR w.r.t. both Delta and Gamma can also be computed by Fourier inversion (Albanese et al.; 2000).

- A Fourier inversion technique can be applied even if the Y_i's are not independent (Glasserman et al.; 2000), but the method is then much slower.
- Only drawback: relatively high "start-up" cost of doing the eigenvalue decomposition.

Moment-Based Methods

cumulants of the P&L-distribution are known:

$$\kappa_r = \frac{1}{2}\sum_i \{(r-1)!\lambda_i^r + r!\delta_i^2\lambda_i^{r-2}\} \qquad (r \geq 2)$$

$$\kappa_1 = \frac{1}{2}\sum_i \lambda_i$$

and can be expressed in terms of Γ, Σ, and Δ avoiding the eigenvalue decomposition:

$$\kappa_r = \frac{1}{2}(r-1)!\mathrm{tr}((\Gamma\Sigma)^r) + \frac{1}{2}r!\Delta^\top\Sigma(\Gamma\Sigma)^{r-2}\Delta \qquad (r \geq 2)$$

Salomon-Stephens approximation

Britton-Jones and Schaefer (1999):

approximate distribution of P (or $-P$) by

$$c_1 w^{c_2} \qquad (w \sim \chi_p^2)$$

c_i such that first three moments are matched.

problem: distribution is bounded below (or above)

Johnson Transformations

Approximate P by matching the first four moments with the moments of $f(X)$, where f is a monotonic transformation (depending on 4 parameters) and $X \sim N(0,1)$.

e.g.

$$f(X) = \sinh\left(\frac{X - \gamma}{\delta}\right)\lambda + \xi$$

(Moments of $f(X)$ are known explicitly.)

Mina and Ulmer (1999): difficult to fit, "not a robust choice"

Gram-Charlier Series

Ansatz for the density:

$$f(x) = \phi(x) \sum_{k=0}^{\infty} c_k H_k(x)$$

where ϕ is the standard normal density and H_k are the *Hermite polynomials*:

$$\phi^{(k)}(x) = (-1)^k H_k(x)\phi(x).$$

The Hermite polynomials are orthogonal w.r.t. the weight function ϕ:

$$\int_{-\infty}^{\infty} H_i(x)H_j(x)\phi(x)dx = j!\delta_{ij}.$$

(Proof: partial integration.)
The Hermite polynomials can be evaluated through the three-term recursion

$$H_{j+1} = xH_j - jH_{j-1}.$$

Proof: induction

$$H_0(x) = 1$$
$$H_1(x) = x$$
$$H_2(x) = x^2 - 1$$
$$H_3(x) = x^3 - 2x$$

This implies the following form for the probability distribution function

$$F(x) = N(x) + \sum_{k=1}^{\infty} c_k (-1)^k \phi^{k-1}(x)$$

and the moment generating function

$$M(t) = e^{\frac{1}{2}t^2} \sum_{k=0}^{\infty} c_k t^k.$$

The Gram-Charlier coefficients can be expressed in terms of the moments:

$$c_k = \sum_{j=0}^{[k/2]} (-1)^j \frac{\mu_{k-2j}}{(k-2j)!j!2^j}.$$

Closely related to the Gram-Charlier series is the *Edgeworth expansion*. Consider a sequence of i.i.d. random variables X_i with cumulants κ_r. Then the cumulant-generating function of $S = \frac{1}{\sqrt{n}}(X_1 +$

$\ldots + X_n - n\kappa_1)$ is

$$nK(t/\sqrt{n}) = \frac{\kappa_2}{2}t^2 + \frac{\kappa_3}{3!}n^{-1/2}t^3 + \frac{\kappa_4}{4!}n^{-1}t^4 \ldots$$

Going to the moment generating function and collecting terms w.r.t. powers of $\sqrt{n}$ instead of powers of t gives the Edgeworth expansion:

$$M(t/\sqrt{n})^n = e^{\frac{1}{2}\kappa_2 t^2} \sum_{k=0}^{\infty} n^{-k/2} P_k(t)$$

where P_k are the *Cramer-Edgeworth polynomials*.

Cornish-Fisher Expansion

(1) Normalize the random variable: $\tilde{P} = (P - \kappa_1)/\sqrt{\kappa_2}$; i.e. $\tilde{\kappa}_1 = 0, \tilde{\kappa}_r = \kappa_r \kappa_2^{-r/2}$

(2) Ansatz:

$$\tilde{q}_\alpha = \sum_{k=0}^{\infty} \frac{d_k}{k!} z_\alpha^k.$$

This implies

$$N(z_\alpha) = \alpha = F(\sum_{k=1}^{\infty} \frac{d_k}{k!} z_\alpha^k).$$

Idea: develop $\tilde{F}$ around z_α and invert.

Up to the fourth cumulant:

$$\tilde{q}_\alpha \approx -\frac{\tilde{\kappa}_3}{6} + (1 - \frac{\tilde{\kappa}_4}{8} + \frac{5\tilde{\kappa}_3}{36})z_\alpha + \frac{\tilde{\kappa}_3}{6} z_\alpha^2 + (\frac{\tilde{\kappa}_4}{24} - \frac{\tilde{\kappa}_3^2}{18})z_\alpha^3$$

Saddle Point Approximations

several ideas combined:

1. exponential tilting
2. Laplace-approximation
3. path-integrals
4. uniform approximations

Quadratic Programming

Britton-Jones and Schaefer (1999): Wilson 1994

$\overline{\mathrm{VaR}_\alpha}$:= worst case in a specific α-confidence region (defined by iso-density lines)

- simply not the same as VaR

! upper bound for the real VaR

The set $\{x \,|\, \Delta^\top x + \frac{1}{2} X^\top \Gamma X\}$ is quadratically bounded, but not necessarily convex.

Monte-Carlo, Quasi-Monte-Carlo

probably only competitive where Fourier inversion methods are infeasible because of large number of risk factors

Conclusion

1. Fourier inversion
   + easy to generalize to other distributions
   + fast in terms of accuracy
   – high "startup" cost (eigenvalue problem)
2. moment-based methods
   + lower startup costs
   i) fixed number of parameters
      i. Johnson distributions
      ii. Salomon-Stephens-approximation
      – somewhat ad-hoc

– fitting may not be robust

ii) series expansions

i. Gram-Charlier series (Edgeworth series is related)

ii. Cornish-Fisher expansion

3. Saddle-point methods

+ can be combined with both Fourier inversion and series expansion techniques

? supposed to drastically speed up convergence

Fourier- and Other Transforms

random variable X with distribution function F and density f

characteristic function:

$$\begin{aligned} \phi_X(t) = \mathrm{E}e^{itX} &= \int_{-\infty}^{\infty} e^{itx} F(dx) \\ &= \int_{-\infty}^{\infty} e^{itx} f(x) dx \end{aligned}$$

- Fourier transform of the density f
- exists for $t \in \mathbb{R}$
- for complex arguments: If $\phi(z)$ exists for some $z \in \mathbb{C}$ then also for $z + a$, for all $a \in \mathbb{R}$.
- If $X \geq x_0$ then ϕ is defined on half-spaces $\{z | \mathrm{Im}(z) \geq c\}$.

Laplace transform:

$$L_X(t) = \mathrm{E}e^{-tX}$$

(often only applied to nonnegative X)

moment generating function:

$$M_X(t) = \mathrm{E}e^{tX}$$

! If $M_X(t)$ exists on an interval $t \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$, then

1. every moment exists
2. M_X is analytic:

$$M_X(t) = \sum_{k=0}^{\infty} \mu_k \frac{t^k}{k!}$$

$$\mu_k = \mathrm{E}X^k = \tfrac{d}{dt} M_X(0).$$

cumulant generating function:

$$K_X(t) = \log E e^{tX}$$

$M_X(t) > 0$ for $t \in \mathbb{R}$, so K_X is analytic if M_X is analytic. The power series coefficients are called *cumulants*:

$$K_X(t) = \sum_{r=1}^{\infty} \kappa_r \frac{t^r}{r!}$$

simple conversions:

$$\phi(-it) = L(-t) = M(t) = e^{K(t)}$$

Cumulants and moments can be expressed in terms of each other:

$$\begin{aligned}
\kappa_1 &= \mu_1 \\
\kappa_2 &= \mu_2 - \mu_1^2 && = \mathrm{E}[(X-\mu_1)^2] =: \sigma^2 \\
\kappa_3 &= \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3 && = \mathrm{E}[(X-\mu_1)^3] \\
\kappa_4 &= \dots && = \mathrm{E}[(X-\mu_1)^4] - 3\sigma^4
\end{aligned}$$

$$\frac{\mu_k}{k!} = \sum_{j=1}^{k} \frac{1}{j!} \sum_{r \in S(j,k)} \prod_{i=1}^{j} \frac{\kappa_{r_i}}{r_i!}$$

$$S(j,k) = \{(r_1,\dots,r_j) \,|\, \sum_{i=1}^{j} r_i = k, r_i \in \{1,\dots,k\}\}$$

multivariate version:

$$\phi_X(t) = \mathrm{E} e^{i\langle t,x\rangle}$$

Example

X Gaussian $N(\mu,\sigma)$:

$$\begin{aligned}
\phi_X(t) &= e^{i\mu t - \sigma^2 t^2/2} \\
K_X(t) &= \mu t + \sigma^2 t^2/2
\end{aligned}$$

multivariate:

$$K_X(t) = \langle \mu, t \rangle + \frac{1}{2}\langle t, \Sigma t \rangle$$

Properties

1. $\phi(0) = 1$, $|\phi(t)| \leq 1$.
2. $\phi(-t) = \overline{\phi(t)}$ for $t \in \mathbb{R}$
3. symmetric distribution implies that $\phi(t)$ is symmetric and real (on $\mathbb{R}$)
4. shifting: $\phi_{X+c}(t) = \mathrm{E}e^{it(X+c)} = e^{ict}\phi_X(t)$
5. scaling: $\phi_{aX}(t) = \mathrm{E}e^{itaX} = \phi_X(at)$
6. convolution: X, Y independent

$$\phi_{X+Y}(t) = \mathrm{E}e^{it(X+Y)} = \mathrm{E}e^{itX}e^{itY} = \phi_X(t)\phi_Y(t)$$

$$(f * g)(x) = \int_{-\infty}^{\infty} f(z)g(x-z)dz$$

$$K_{X+Y}(t) = K_X(t) + K_Y(t)$$

7. inversion (and uniqueness (Levy 1925))

$$
\begin{aligned}
f_X(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}\phi_X(t)dt \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ixt}M_X(it)dt \\
&= \frac{1}{2\pi}\int_{-i\infty}^{+i\infty} e^{-xt}M_X(t)dt \\
F_X(b) - F_X(a) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-itb}-e^{-ita}}{-it}\phi_X(t)dt
\end{aligned}
$$

(F_X is the version with $F_X(x) = (F_X(x+) + F_X(x-))/2.$)

8. differentiation

$$(\Phi f)'(t) = \int_{-\infty}^{\infty} e^{itx} ixf(x)dx = i(\Phi xf(x))(t)$$

9. Riemann-Lebesgue-Lemma:
   If X has a density, then $\lim_{|t|\to\infty}\phi_X(t) = 0$.

Discrete Fourier Transform

$$(\mathrm{DFT}h)(n) := H_n = \sum_{k=0}^{N-1} h_k e^{2\pi ikn/N}$$

! H_n is periodic with period N

inversion:

$$h_k = \frac{1}{N} \sum_{n=0}^{N-1} H_n e^{-2\pi i k n/N}$$

distribution on a regular grid:

$$P\{X = x_0 + \Delta k/N\} = h_k$$

$$\begin{aligned} \phi_X(t) &= \mathrm{E} e^{itX} = \sum_{k=0}^{N-1} h_k e^{it(x_0 + \Delta k/N)} \\ &= e^{i x_0 t} (\mathrm{DFT} h)(\frac{\Delta t}{2\pi}) \end{aligned}$$

Fast Fourier Transform

first glance: matrix-vector multiplication $\rightarrow O(N^2)$ operations

$$
\begin{aligned}
H_n &= \sum_{k=0}^{N-1} h_k e^{2\pi i k n/N} \\
&= \sum_{k=0}^{N/2-1} h_{2k} e^{i2\pi 2kn/N} + \sum_{k=0}^{N/2-1} h_{2k+1} e^{2\pi i(2k+1)n/N} \\
&= (\mathrm{DFT} h^{\mathrm{even}})(n) + e^{2\pi i n/N} (\mathrm{DFT} h^{\mathrm{odd}})(n)
\end{aligned}
$$

$\rightarrow O(N \log_2 N)$ operations

implementation:

- implemented in most interactive packages like *S-Plus*...
- many tricks possible
- use the "Fastest Fourier Transform in the West" as a portable library
- Pentium II 300MHz, 65MFlops, 5 $N \log_2 N$ operations:

N	time in seconds
$2^{16} = 65536$	0.08
$2^{20} = 1048580$	1.60

Further Reading

For properties on characteristic functions consult any text book on probability and statistics. For the fast Fourier transform see for example (Press et al.; 1992, p.500). For implementation issues consult (Frigo and Johnson; 1998).

References

Albanese, C., Jackson, K. and Wiberg, P. (2000). Fast convolution method for VaR and VaR gradients, http://www.math-point.com/fconv.ps.

Anderson, E., Bai, Z., Bischof, C., Blackford, S., Demmel, J., Dongarra, J., Croz, J. D., Greenbaum, A., Hammarling, S., McKenney, A. and Sorensen, D. (1999). *LAPACK Users' Guide*, third edn, SIAM. http://www.netlib.org/lapack/lug/.

Artzner, P., Delbaen, F., Eber, J.-M. and Heath, D. (1998). Coherent measures of risk, http://www.math.ethz.ch/~delbaen/CoherentMF.pdf. Working paper in several versions since 1996.

Basel Committee on Banking Supervision (1988). International convergence of capital measurement and capital standards, http://www.bis.org/publ/bcbs04a.pdf.

Basel Committee on Banking Supervision (1993). The supervisory treatment of market risks, Bank of International Settlement. for comment.

Basel Committee on Banking Supervision (1995). An internal model-based approach to market risk capital requirements, http://www.bis.org/publ/bcbsc224.pdf.

Basel Committee on Banking Supervision (1996). Amendment to the capital accord to incorporate market risks, http://www.bis.org/publ/bcbs24.pdf.

Basel Committee on Banking Supervision (1998a). Amendment to the capital accord to incorporate market risks, January 1996, updated April 1998, http://www.bis.org/publ/bcbsc222.pdf.

Basel Committee on Banking Supervision (1998b). International convergence of capital measurement and capital standards, July 1988, updated to April 1998, http://www.bis.org/publ/bcbsc111.pdf.

Basel Committee on Banking Supervision (1999a). Credit risk modelling: Current practices and applications, http://www.bis.org/publ/bcbs49.pdf.

Basel Committee on Banking Supervision (1999b). A new capital adequacy framework, http://www.bis.org/publ/bcbs50.pdf.

Basel Committee on Banking Supervision (2000a). History of the Basel Committee and its membership, http://www.bis.org/publ/bcbsc101.pdf.

Basel Committee on Banking Supervision (2000b). A new capital adequacy framework: Pillar 3 - market discipline, http://www.bis.org/publ/bcbs65.pdf.

Basel Committee on Banking Supervision (2000c). Principles for the management of credit risk, http://www.bis.org/publ/bcbs75.pdf.

Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities, *Journal of Political Economy* **81**: 637–654.

Britton-Jones, M. and Schaefer, S. (1999). Non-linear Value-at-Risk, *European Finance Review* **2**: 161–187.

Bundesaufsichtsamt für Kreditwesen (2000). Grundsatz I über die Eigenmittel der Institute, Bekanntmachung vom 29. Oktober 1997, zuletzt geändert durch die Bekanntmachung vom 20. Juli 2000, http://www.bakred.de/texte/bekannt/g1_20070.htm.

Bundesaufsichtsamt für Kreditwesen (1997a). Bekanntmachung über die Änderung und Ergänzung der Grundsätze über das Eigenkapital und die Liquidität der Kreditinstitute vom 29. Oktober 1997 (Grundsatz I), was available as http://www.bakred.de/texte/bekannt/grds1.htm, now superseded by http://www.bakred.de/texte/bekannt/g1_20070.htm.

Bundesaufsichtsamt für Kreditwesen (1997b). Erläuterungen zur Bekanntmachung über die Änderung und Ergänzung der Grundsätze über das Eigenkapital und die Liquidität der Kreditinstitute vom 29. Oktober 1997 (Grundsatz I), http://www.bakred.de/texte/bekannt/gs1_erl.pdf.

Chicago Board of Trade (2000). Chronological history, http://www.cbot.com/.

Chicago Board Options Exchange (2000). Some history ..., http://www.cboe.com/exchange/cboehistory.htm.

Chicago Mercantile Exchange (2000a). CME history of innovation, http://www.cme.com/exchange/history.html.

Chicago Mercantile Exchange (2000b). SPAN: The Standard Portfolio Analysis of Risk system, http://www.cme.com/span/.

Cox, J. and Ross, A. (1976). The valuation of options for alternative stochastic processes, *Journal of Financial Economics* **3**(2): 145–166.

Davies, R. (2000). Classic financial scandals, http://www.ex.ac.uk/~RDavies/arian/scandals/classic.html. link list to documents about the BCCI, Barings, Daiwa, etc. cases.

Deutsche Bundesbank (1999). Gesetz über das Kreditwesen - KWG, http://www.bakred.de/texte/gesetz/kwg_9912.htm. Lesefassung vom BAKred.

Dowd, K. (1998). *Beyond Value at Risk: The New Science of Risk Management*, Wiley Frontiers in Finance, Wiley.

Embrechts, P., Klüppelberg, C. and Mikosch, T. (1997). *Modelling extremal events*, Springer-Verlag, Berlin.

EU Council (1993). Council Directive 93/6/EEC of 15 March 1993 on the capital adequacy of investments firms and credit institutions, http://europa.eu.int/eur-lex/en/lif/dat/1993/en_393L0006.html. Document 393L0006.

EU Council (1998a). Directive 98/31/EC of the European Parliament and of the Council of 22 June 1998 amending Council Directive 93/6/EEC on the capital adequacy of investment firms and credit institutions, http://europa.eu.int/eur-lex/en/lif/dat/1998/en_398L0031.html. Document 398L0031.

EU Council (1998b). Directive 98/33/EC of the European Parliament and of the Council of 22 June 1998 amending amending Article 12 of Council Directive 77/780/EEC on the taking up and pursuit of the business of credit institutions, Articles 2, 5, 6, 7, 8 of and Annexes II and III to Council Directive 89/647/EEC on a solvency ratio for credit institutions and Article 2 of and Annex II to Council Directive 93/6/EEC on the capital adequacy of investment firms and credit institutions, http://europa.eu.int/eur-lex/en/lif/dat/1998/en_398L0033.html. Document 398L0033.

European Commission (1999). A review of regulatory capital requirements for EU credit institutions and investment firms, http://europa.eu.int/comm/internal_market/en/finances/banks/capaden.pdf. Consultation Document.

European Commission (2000). Institutional arrangements for the regulation and supervision of the financial sector, http://europa.eu.int/comm/internal_market/en/finances/banks/arrange.pdf. Overview Document.

Frigo, M. and Johnson, S. G. (1998). FFTW: An adaptive software architecture for the FFT, *ICASSP conference proceedings*, Vol. 3, p. 1381. http://www.fftw.org/.

Gaumert, U. and Stahl, G. (2000). Interne Modelle, Diskussionspapier BAKred.

Glasserman, P., Heidelberger, P. and Shahabuddin, P. (2000). Portfolio value-at-risk with heavy-tailed risk factors, http://www.columbia.edu/cu/business/wp/00/pw-00-06.pdf.

Härdle, W. and Stahl, G. (1999). Backtesting beyond VaR, http://sfb.wiwi.hu-berlin.de/papers/1999/dpsfb1999105.ps.Z. Discussion Paper 99-105, Sonderforschungbereich 373, Humboldt-Universität Berlin.

Harrison, J. M. and Kreps, D. (1979). Martingales and arbitrage in multiperiod security markets, *Journal of Economic Theory* **20**: 381–408.

Hodges, S. D. and Neuberger, A. (1989). Optimal replication of contingent claims under transaction costs, *Rev. Futures Markets* **8**: 222–239.

Jaschke, S. and Küchler, U. (1999). Coherent risk measures, valuation bounds, and (μ, ρ)-portfolio optimization, Discussion Paper 99-64, SFB 373, Humboldt-Universität zu Berlin, forthcoming in Finance and Stochastics. http://www.jaschke.net/papers/crm.pdf.

Jorion, P. (1997). *Value at Risk: The New Benchmark for Controlling Market Risk*, McGraw-Hill, New York.

Jorion, P. (2000). *Value at Risk: The New Benchmark for Managing Financial Risk*, McGraw-Hill, New York.

Kupiec, P. H. and O'Brien, J. M. (1997). The pre-commitment approach: Using incentives to set market risk capital requirements, http://www.federalreserve.gov/pubs/feds/1997/199714/199714pap.ps.

Longerstaey, J. (1996). RiskMetrics technical document, *Technical Report fourth edition*, J.P.Morgan. originally from http://www.jpmorgan.com/RiskManagement/RiskMetrics/, now http://www.riskmetrics.com.

Markowitz, H. M. (1959). *Portfolio selection: Efficient diversification of investments*, John Wiley & Sons Inc., New York.

Mina, J. (1999). Improved cash flow map, http://www.riskmetrics.com/research/working/cashflowmap.pdf. The RiskMetrics Group.

Mina, J. and Ulmer, A. (1999). Delta-gamma four ways, http://www.riskmetrics.com.

Press, W. H., Teukolsky, S. A., Vetterling, W. T. and Flannery, B. P. (1992). *Numerical Recipes in C*, second edn, Cambridge University Press.

Taleb, N. (2000). Fooled by randomness: If you're so rich why aren't you so smart?, http://pw1.netcom.com/~ntaleb/trader.htm. unfinished book, some sections online.

Traber, U. (2000). Die Prüfung und Zulassung bankinterner Marktrisikomodelle nach § 32 Grundsatz I, *WM* **20**: 992–1000.

Traub, J. F. and Werschulz, A. G. (1998). *Complexity and Information*, Lezioni Lincee, Cambridge University Press.