Chapter 3

Dynamic optimization and utility functions

This chapter offers a brief account of one of the leading approaches to solving dynamic optimization problems and how the so called Euler equation can be derived. We will apply the Lagrange multiplier method (applying the Kuhn–Tucker Theorem) and show how the Euler equation is derived within both OLG and Ramsey models. In addition, we briefly discuss some specific utility functions that we will use in the next chapter when studying consumption theory. For a more detailed discussion about dynamic optimization and other methods to solving optimization problems the reader should consult for example Dixit (1990), Sydsæter och Hammond (1995) or Obstfeld and Rogoff (1996).

3.1 The Euler equation within a 2–period model

Let us consider a 2 period model with one consumer good. Assume also that the utility function is intertemporally additive, i.e., the rate of substitution between consumption on any two dates is independent of consumption on any third date. Thus, we rule out any intertemporal consumption dependencies such as habit formation. The individual maximizes the utility of consumption over the two dates given labor income:

$$U = U\left(C_1\right) + \beta U\left(C_2\right) \qquad 0 < \beta < 1 \tag{3.1}$$

where c is consumption and $\beta = 1/\left(1+\theta\right)$ is a measure of the individual's impatience to consume, where θ is the the subjective rate of time preference. The utility function $U\left(C\right)$ is increasing in consumption and strictly concave: $U'\left(C\right) > 0$, $U''\left(C\right) < 0$. In addition we assume that $\lim_{C\to 0} U'\left(C\right) = \infty$ which implies that the individual always desire at least a little consumption in every period, i.e., consumption is always positive so we don't have to add the constraint that $C_i \geq 0$ to the maximization problem.

The individual maximizes the utility in (3.1) with respect to the budget constraints

$$C_1 = Y_1 - B_1 \tag{3.2}$$

and

$$C_2 = Y_2 + (1 + r) B_1 \tag{3.3}$$

where Y is labor income and B is the value of net assets at the end of period t (savings). Use equation (3.2) to solve for B_1 and insert into (3.3) such that the budget restriction can be written as

$$C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r}$$

which states that the present value of consumption is equal to the present value of labor income. Note that we rule out inheritance. To find a solution to the optimization problem, i.e., determine C_1 and C_2, we form the Lagrangian

$$L = U(C_1) + \beta U(C_2) - \lambda \left[C_1 + \frac{C_2}{1+r} - Y_1 - \frac{Y_2}{1+r} \right].$$

and then differentiating it partially with respect to consumption in the two periods. The first order conditions are

$$\frac{\partial L}{\partial C_1} = U'(C_1) - \lambda = 0$$

and

$$\frac{\partial L}{\partial C_2} = \beta U'(C_2) - \lambda \left[\frac{1}{1+r} \right] = 0.$$

Eliminate the Lagrange multiplier by taking the ratio of these two first order conditions such that

$$U'(C_1) = (1 + r) \beta U'(C_2). \tag{3.4}$$

This is the intertemporal Euler equation which is the necessary condition for optimum. The Euler equation has a simple interpretation: at a utility maximum, the individual cannot gain from redistributing consumption between periods. A one unit reduction in first period consumption lowers the utility in this period by $U'(C_1)$. The consumption unit which is now saved in the first period can be converted into $1 + r$ units of second period consumption that raise second period utility by $(1 + r) \beta U'(C_2)$ units. The Euler equation states that these two quantities are equal at an optimum.

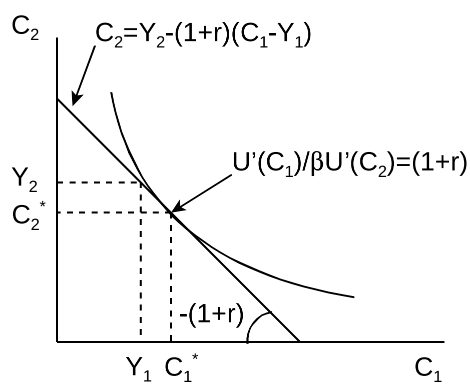
The Euler equation (3.4) can also be written in the following way

$$\frac{\beta U'(C_2)}{U'(C_1)} = \frac{1}{1+r}$$

where the LHS is the individual's marginal rate of substitution between consumption in the two periods whereas the RHS is the price of future consumption in terms of present consumption. We can also illustrate the Euler equation in a graph, see figure 3.1. Point A in Figure 3.1 is the optimum where the slope of the indifference curve is equal to the slope of the budget line, i.e., where

$$\frac{U'(C_1)}{\beta U'(C_2)} = (1 + r).$$

Figure 3.1: Consumption over time and the Euler equation.



Assume now that $\beta(1+r)=1$. Under this assumption, the Euler equation reduces to

$$U'(C_1)=U'(C_2).$$

This implies that the individual prefer to smooth consumption over time such that $C_1 = C_2 = \bar{C}$.[1] If we insert this solution into the budget restriction we obtain

$$\bar{C}=\frac{[(1+r)Y_1+Y_2]}{2+r}$$

which, if we assume that $r=0$, is

$$\bar{C}=\frac{Y_1+Y_2}{2}$$

implying that consumption in the two periods is equal to the average of labor income. Consequently, the optimum must then be located on a 45^0–degree line as is shown in Figure 3.2. This is the permanent income theory.

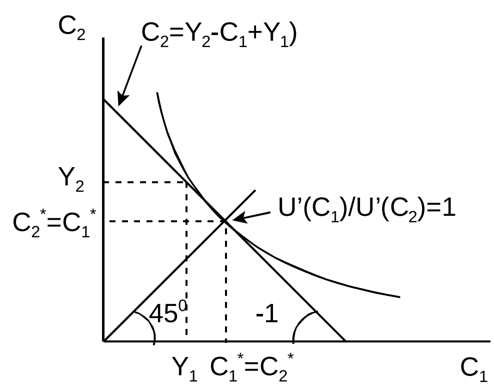
3.2 The Euler equation in a Ramsey model

Let us now extend the number of periods to T periods. In this case the individual seeks to maximize a sequence of consumption over all time periods, i.e., c_t, c_{t+1}, ... that maximizes utility. The optimization problem can then be formulated as

$$\max\sum_{s=t}^{t+T}\beta^{s-t}U(C_s)$$

[1]The reason is that in this case the expected marginal utility from consumption in period two is equal to the present marginal utility from consumption.

Figure 3.2: Consumption over time when $\beta = 1$ and $r = 0$.



with respect to

$$\sum_{s=t}^{t+T}\left(\frac{1}{1+r}\right)^{s-t}C_s = (1+r)\,B_t + \sum_{s=t}^{t+T}\left(\frac{1}{1+r}\right)^{s-t}Y_s.$$

The Lagrangian is therefore

$$L = \sum_{s=t}^{t+T}\beta^{s-t}U\left(C_s\right) - \lambda\left[\sum_{s=t}^{t+T}\left(\frac{1}{1+r}\right)^{s-t}C_s - (1+r)\,B_t + \sum_{s=t}^{t+T}\left(\frac{1}{1+r}\right)^{s-t}Y_s\right]$$

and the first order conditions with respect to consumption at time s is

$$\frac{\partial L}{\partial C_s} = \beta^{s-t}\frac{\partial U}{\partial C_s} - \lambda\left(\frac{1}{1+r}\right)^{s-t} = 0$$

and for time period $s+1$

$$\frac{\partial L}{\partial C_{s+1}} = \beta^{s+1-t}\frac{\partial U}{\partial C_{s+1}} - \lambda\left(\frac{1}{1+r}\right)^{s+1-t} = 0.$$

Eliminating the Lagrange multiplier as we did above yields

$$U'\left(C_s\right) = (1+r)\,\beta U'\left(C_{s+1}\right) \tag{3.5}$$

which is identical to the Euler equation within the 2–period model above. Note also that if we substitute the Euler equation forward we find that for any $T \geq 0$

$$U'\left(C_t\right) = (1+r)^T\,\beta^T U'\left(C_{t+T}\right) \tag{3.6}$$

From this follows that as time goes to infinity, the Euler equation can still be formulated as (3.5) and (3.6). If we in addition also assume uncertainty so that the individual use all available information at time t, the optimization problem can be formulated as follows

$$\max \quad \mathrm{E}\left[\sum_{s=t}^{t+T}\beta^{s-t}U\left(C_s\right)|t\right]$$

with respect to

$$\sum_{s=t}^{t+T}\left(\frac{1}{1+r}\right)^{s-t}C_s=\sum_{s=t}^{t+T}\left(\frac{1}{1+r}\right)^{s-t}Y_s.$$

The Euler equation can then be written as (given that the interest rate r is constant and known at time t)

$$U'\left(C_t\right)=\left(1+r\right)\beta\mathrm{E}\left[U'\left(C_{t+1}\right)|t\right]$$

which is identical to the earlier Euler equations except for the inclusion of the expectations operator.

Finally, let us assume that there are $i\in I$ assets in the economy instead of just one as we assumed in the models above. The rate of return from an asset i at time $t+1$ is r_{t+1}^i. Assume also that this return is uncertain. In this case we have one Euler equation for each asset i given by

$$U'\left(C_t\right)=\beta\mathrm{E}\left[\left(1+r_{t+1}^i\right)U'\left(C_{t+1}\right)|t\right]\qquad \text{for all } i\in I.$$

3.3 Utility functions

In the models analyzed above, we did not specify the utility functions. Let us in this section define a number of utility functions that we will use in the next chapter in the context of consumption theory, i.e., constant relative risk aversion, constant absolute risk aversion and quadratic utility functions.

3.3.1 Constant relative risk aversion (CRRA)

The constant relative risk aversion (CRRA) utility function (see Romer (2001, pp. 48) is given by

$$\begin{aligned} U\left(C\right) &= \frac{C^{1-\gamma}}{1-\gamma} \quad \text{for } \gamma>0,\gamma\neq 1,\\ &= \ln C \quad \text{for } \gamma=1 \end{aligned}$$

where $1/\gamma$ is the intertemporal substitution elasticity between consumption in any two periods, i.e., it measures the willingness to substitute consumption between different periods. The smaller γ (the larger $1/\gamma$), the more willing is the household to substitute consumption over time. Note also that γ is the coefficient of relative risk aversion.[2] Since the coefficient of relative risk aversion is constant, this utility function is known as constant relative risk aversion (CRRA) utility.

There are three other properties that are important. First, the CRRA utility function is increasing in $C^{1-\gamma}$ if $\gamma < 1$ but decreasing if $\gamma > 1$. Therefore, dividing by $1-\gamma$ ensures that the marginal utility is positive for all values of γ. Second, if $\gamma \to 1$, the utility function converges to $\ln C_t$.[3] Third, $U'''(C) > 0$, implying a positive motive for precautionary saving. Therefore, we often use this utility function when studying savings behavior.

3.3.2 Constant absolute risk aversion (CARA)

The other class of utility functions often used in intertemporal models is the following exponential utility function

$$U(C) = -\frac{1}{\alpha}\exp(-\alpha C) \qquad \alpha > 0$$

which is known as the constant absolute risk aversion (CARA) utility. For this utility function, $U'(C) = \exp(-\alpha C)$ and $U''(C) = -\alpha \exp(-\alpha C)$. Applying the definition of the coefficient of constant absolute risk aversion $-U''(C)/U'(C)$ we find that α is the coefficient of absolute risk aversion.[4] Constant absolute risk aversion is usually viewed as a less credible description of risk aversion compared to relative risk aversion, but the CARA–utility function is often a more convenient specification analytically. As is the case with CRRA utility, the CARA function implies a positive motive for precautionary saving.

3.3.3 Quadratic utility function

The third class of utility functions we will use below is the quadratic utility function

$$U = C - \frac{a}{2}C^2, \qquad a > 0$$

[2]This can be seen if we use the Arrow–Pratt definition of the coefficient of relative risk averision:

$$-\frac{CU''(C)}{U'(C)}.$$

[3]To show that the utility function converges to logarithmic as $\gamma \to \infty$ we make use of L'Hospital's rule. As $\gamma \to \infty$, the numerator and denominator of the function both approach zero. Differentiate both the numerator and the denominator with respect to γ and then take the limit of the derivatives' ratio as $\gamma \to \infty$ we find that the utility function converges to $\ln C$.

[4]Note that the coefficient for relative risk aversion is equal to αC such that the degree of risk aversion is increasing in consumption.

where the marginal utility is linear, see Romer (2001, pp. 337).[5] As for the earlier two utility functions, $U'(C) = 1 - aC > 0$ (assuming that the parameter a is sufficiently small in relation to C) while $U''(C) = -a < 0$. Since $U''' = 0$ there is no motive for precautionary saving. This utility function is mainly used in the context of permanent income and life cycle hypotheses.

[5] Note that we in this case consider only the case when $C \in \left[0, \frac{1}{a}\right]$. The reason is that the utility function is not strictly concave if $C < \frac{1}{a}$.