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# A comparative anatomy of credit risk models

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## Abstract

Within the past two years, important advances have been made in modeling credit risk at the portfolio level. Practitioners and policy makers have invested in implementing and exploring a variety of new models individually. Less progress has been made, however, with comparative analyses. Direct comparison often is not straightforward, because the different models may be presented within rather different mathematical frameworks. This paper offers a comparative anatomy of two especially influential benchmarks for credit risk models, the RiskMetrics Group's CreditMetrics and Credit Suisse Financial Product's CreditRisk<sup>+</sup>. We show that, despite differences on the surface, the underlying mathematical structures are similar. The structural parallels provide intuition for the relationship between the two models and allow us to describe quite precisely where the models differ in functional form, distributional assumptions, and reliance on approximation formulae. We then design simulation exercises which evaluate the effect of each of these differences individually. © 2000 Elsevier Science B.V. All rights reserved.

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## 1. Introduction

Over the past decade, financial institutions have developed and implemented a variety of sophisticated models of value-at-risk for market risk in trading portfolios. These models have gained acceptance not only among senior bank

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managers, but also in amendments to the international bank regulatory framework. Much more recently, important advances have been made in modeling credit risk in lending portfolios. The new models are designed to quantify credit risk on a portfolio basis, and thus have application in control of risk concentration, evaluation of return on capital at the customer level, and more active management of credit portfolios. Future generations of today's models may one day become the foundation for measurement of regulatory capital adequacy.

Two of the models, the RiskMetrics Group's CreditMetrics and Credit Suisse Financial Product's CreditRisk<sup>+</sup>, have been released freely to the public since 1997 and have quickly become influential benchmarks. Practitioners and policy makers have invested in implementing and exploring each of the models individually, but have made less progress with comparative analyses. The two models are intended to measure the same risks, but impose different restrictions and distributional assumptions, and suggest different techniques for calibration and solution. Thus, given the same portfolio of credit exposures, the two models will, in general, yield differing evaluations of credit risk. Determining which features of the models account for differences in output would allow us a better understanding of the sensitivity of the models to the particular assumptions they employ.

Direct comparison of the models has so far been limited, in large part, because the two models are presented within rather different mathematical frameworks. The CreditMetrics model is familiar to econometricians as an ordered probit model. Credit events are driven by movements in underlying unobserved latent variables. The latent variables are assumed to depend on external "risk factors". Common dependence on the same risk factors gives rise to correlations in credit events across obligors. The CreditRisk<sup>+</sup> model is based instead on insurance industry models of event risk. Instead of a latent variable, each obligor has a default probability. The default probabilities are not constant over time, but rather increase or decrease in response to background systemic factors. To the extent that two obligors are sensitive to the same set of background factors, their default probabilities will move together. These comovements in probability give rise to correlations in defaults. CreditMetrics and CreditRisk<sup>+</sup> may serve essentially the same function, but they appear to be constructed quite differently.

This paper offers a comparative anatomy of CreditMetrics and CreditRisk<sup>+</sup>. We show that, despite differences on the surface, the underlying probabilistic structures are similar. Understanding the structural parallels helps to develop intuition for the relationship between the two models. More importantly, it allows us to describe quite precisely where the models differ in functional form, distributional assumptions, and reliance on approximation formulae. We then design simulation exercises which evaluate the effect of each of these differences individually.

We proceed as follows. Section 2 presents a summary of the CreditRisk<sup>+</sup> model and introduces a restricted version of CreditMetrics. The restrictions are imposed to facilitate direct comparison of CreditMetrics and CreditRisk<sup>+</sup>. While some of the richness of the full CreditMetrics implementation is sacrificed, the essential mathematical characteristics of the model are preserved. Our method of comparative anatomy is developed in Section 3. We show how the restricted version of CreditMetrics can be run through the mathematical machinery of CreditRisk<sup>+</sup>, and vice versa.

Comparative simulations are developed in Section 4. Care is taken to construct portfolios with quality and loan size distributions similar to real bank portfolios, and to calibrate correlation parameters in the two models in an empirically plausible and mutually consistent manner. The robustness of the conclusions of Section 4 to our methods of portfolio construction and parameter calibration are explored in Section 5. An especially striking result from the simulations is the sensitivity of CreditRisk<sup>+</sup> results to the shape of the distribution of a systemic risk factor. The reasons for and consequences of this sensitivity are explored in Section 6. We conclude with a summary of the main results of the simulations.

## 2. Summary of the models

This section offers an introduction to CreditRisk<sup>+</sup> and CreditMetrics. The discussion of CreditRisk<sup>+</sup> merely summarizes the derivation presented in CSFP (1997, Appendix A). Our presentation of CreditMetrics sets forth a restricted version of the full model described in the CreditMetrics Technical Document (Gupton et al., 1997). Our choice of notation is intended to facilitate comparison of the models, and may differ considerably from what is used in the original manuals.

### 2.1. Summary of CreditRisk<sup>+</sup>

CreditRisk<sup>+</sup> is a model of *default risk*. Each obligor has only two possible end-of-period states, default and non-default. In the event of default, the lender suffers a loss of fixed size, this is the lender's *exposure* to the obligor. The distributional assumptions and functional forms imposed by CreditRisk<sup>+</sup> allow the distribution of total portfolio losses to be calculated in a convenient analytic form.

Default correlations in CreditRisk<sup>+</sup> are assumed to be driven entirely by a vector of  $K$  "risk factors"  $x = (x_1, \dots, x_K)$ . Conditional on  $x$ , defaults of individual obligors are assumed to be independently distributed Bernoulli draws. The conditional probability  $p_i(x)$  of drawing a default for obligor  $i$  is a function of the rating grade  $\zeta(i)$  of obligor  $i$ , the realization of risk factors  $x$ ,

and the vector of “factor loadings” ( $w_{i1}, \dots, w_{iK}$ ) which measure the sensitivity of obligor  $i$  to each of the risk factors. CreditRisk<sup>+</sup> specifies this function as

$$p_i(x) = \bar{p}_{\zeta(i)} \left( \sum_{k=1}^K x_k w_{ik} \right), \quad (1)$$

where  $\bar{p}_{\zeta}$  is the unconditional default probability for a grade  $\zeta$  obligor, and the  $x$  are positive-valued with mean one. The intuition behind this specification is that the risk factors  $x$  serve to “scale up” or “scale down” the unconditional  $\bar{p}_{\zeta}$ . A high draw of  $x_k$  (over one) increases the probability of default for each obligor in proportion to the obligor’s weight  $w_{ik}$  on that risk factor, a low draw of  $x_k$  (under one) scales down all default probabilities. The weights  $w_{ik}$  are required to sum to one for each obligor, which guarantees that  $E[p_i(x)] = \bar{p}_{\zeta(i)}$ .

Rather than calculating the distribution of defaults directly, CreditRisk<sup>+</sup> calculates the *probability generating function* (“pgf”) for defaults. The pgf  $\mathcal{F}_{\kappa}(z)$  of a discrete random variable  $\kappa$  is a function of an auxiliary variable  $z$  such that the probability that  $\kappa = n$  is given by the coefficient on  $z^n$  in the polynomial expansion of  $\mathcal{F}_{\kappa}(z)$ . The pgf has two especially useful properties:<sup>1</sup>

- If  $\kappa_1$  and  $\kappa_2$  are independent random variables, then the pgf of the sum  $\kappa_1 + \kappa_2$  is equal to the product of the two pgfs.
- If  $\mathcal{F}_{\kappa}(z|x)$  is the pgf of  $\kappa$  conditional on  $x$ , and  $x$  has distribution function  $H(x)$ , then the unconditional pgf is simply  $\mathcal{F}_{\kappa}(z) = \int_x \mathcal{F}_{\kappa}(z|x) dH(x)$ .

We first derive the conditional pgf  $\mathcal{F}(z|x)$  for the total number of defaults in the portfolio given realization  $x$  of the risk factors. For a single obligor  $i$ , this is the Bernoulli( $p_i(x)$ ) pgf:

$$\mathcal{F}_i(z|x) = (1 - p_i(x) + p_i(x)z) = (1 + p_i(x)(z - 1)). \quad (2)$$

Using the approximation formula  $\log(1 + y) \approx y$  for  $y \approx 0$ , we can write

$$\mathcal{F}_i(z|x) = \exp(\log(1 + p_i(x)(z - 1))) \approx \exp(p_i(x)(z - 1)). \quad (3)$$

We refer to this step as the “Poisson approximation” because the expression on the right-hand side is the pgf for a random variable distributed  $\text{Poisson}(p_i(x))$ . The intuition is that, as long as  $p_i(x)$  is small, we can ignore the constraint that a single obligor can default only once, and represent its default event as a Poisson random variable rather than as a Bernoulli. The exponential form of the Poisson pgf is essential to the computational facility of the model.

Conditional on  $x$ , default events are independent across obligors, so the pgf of the sum of obligor defaults is the product of the individual pgfs:

<sup>1</sup> See Johnson and Kotz (1969, Section 2.2) for further discussion of probability generating functions.

$$\mathcal{F}(z|x) = \prod_i \mathcal{F}_i(z|x) \approx \prod_i \exp(p_i(x)(z - 1)) = \exp(\mu(x)(z - 1)), \tag{4}$$

where  $\mu(x) \equiv \sum_i p_i(x)$ .

To get the unconditional probability generating function  $\mathcal{F}(z)$ , we integrate out the  $x$ . The risk factors in CreditRisk<sup>+</sup> are assumed to be independent gamma-distributed random variables with mean one and variance  $\sigma_k^2$ ,  $k = 1, \dots, K$ .<sup>2</sup> See Appendix A on the properties and parameterization of the gamma distribution. It is straightforward to show that

$$\mathcal{F}(z) = \prod_{k=1}^K \left( \frac{1 - \delta_k}{1 - \delta_k z} \right)^{1/\sigma_k^2} \tag{5}$$

where  $\delta_k \equiv \frac{\sigma_k^2 \mu_k}{1 + \sigma_k^2 \mu_k}$  and  $\mu_k \equiv \sum_i w_{ik} \bar{p}_{\zeta(i)}$ .

The form of this pgf shows that the total number of defaults in the portfolio is a sum of  $K$  independent negative binomial variables.

The final step in CreditRisk<sup>+</sup> is to obtain the probability generating function  $\mathcal{G}(z)$  for losses. Assume loss given default is a constant fraction  $\lambda$  of loan size. Let  $L_i$  denote the loan size for obligor  $i$ . In order to retain the computational advantages of the discrete model, we need to express the loss exposure amounts  $\lambda L_i$  as integer multiples of a fixed unit of loss (e.g., one million dollars). The base unit of loss is denoted  $v_0$  and its integer multiples are called “standardized exposure” levels. The standardized exposure for obligor  $i$ , denoted  $v(i)$ , is equal to  $\lambda L_i / v_0$  rounded to the nearest integer.

Let  $\mathcal{G}_i$  denote the probability generating function for losses on obligor  $i$ . The probability of a loss of  $v(i)$  units on a portfolio consisting only of obligor  $i$  must equal the probability that  $i$  defaults, so  $\mathcal{G}_i(z|x) = \mathcal{F}_i(z^{v(i)}|x)$ . We use the conditional independence of the defaults to obtain the conditional pgf for losses in the entire portfolio as

$$\mathcal{G}(z|x) = \prod_i \mathcal{G}_i(z|x) = \exp \left( \sum_{k=1}^K x_k \sum_i \bar{p}_{\zeta(i)} w_{ik} (z^{v(i)} - 1) \right). \tag{6}$$

As before, we integrate out the  $x$  and rearrange to arrive at

$$\mathcal{G}(z) = \prod_{k=1}^K \left( \frac{1 - \delta_k}{1 - \delta_k \mathcal{P}_k(z)} \right)^{1/\sigma_k^2} \quad \text{where} \quad \mathcal{P}_k(z) \equiv \frac{1}{\mu_k} \sum_i w_{ik} \bar{p}_{\zeta(i)} z^{v(i)} \tag{7}$$

and  $\delta_k$  and  $\mu_k$  are as defined in Eq. (5).

<sup>2</sup> This is a variant on the presentation in the CreditRisk<sup>+</sup> manual, in which  $x_k$  has mean  $\mu_k$  and variance  $\sigma_k^2$ , and the conditional probabilities are given by  $p_i(x) = \bar{p}_{\zeta(i)} (\sum w_{ik} (x_k / \mu_k))$ . In our presentation, the constants  $1/\mu_k$  are absorbed into the normalized  $x_k$  without any loss of generality.

The unconditional probability that there will be  $n$  units of  $v_0$  loss in the total portfolio is given by the coefficient on  $z^n$  in the Taylor series expansion of  $\mathcal{G}(z)$ . The CreditRisk<sup>+</sup> manual (Section A.10) provides the recurrence relation used to calculate these coefficients.

## 2.2. A restricted version of CreditMetrics

The CreditMetrics model for credit events is familiar to economists as an ordered probit. Associated with obligor  $i$  is an unobserved latent random variable  $y_i$ . The state of obligor  $i$  at the risk-horizon depends on the location of  $y_i$  relative to a set of “cut-off” values. In the full version of the model, the cut-offs divide the real number line into “bins” for each end-of-period rating grade. CreditMetrics thereby captures not only defaults, but migrations across non-default grades as well. Given a set of forward credit spreads for each grade, CreditMetrics can then estimate a distribution over the change in mark-to-market value attributable to portfolio credit risk.

In this section, we present a restricted version of CreditMetrics. To allow more direct comparison with CreditRisk<sup>+</sup>, we restrict the set of outcomes to two states, default and non-default. In the event of default, we assume loss is a fixed fraction  $\lambda$  of the face value. This represents a second significant simplification of the full CreditMetrics implementation, which allows idiosyncratic risk in recoveries.<sup>3</sup> In the non-default state, the loan retains its book value. Thus, our restricted version of CreditMetrics is a model of book value losses, rather than of changes in market value. In the discussion below, the restricted CreditMetrics will be designated as “CM2S” (“CreditMetrics two-state”) whenever distinction from the full CreditMetrics model needs emphasis.

The latent variables  $y_i$  are taken to be linear functions of risk factors  $x$  and idiosyncratic effects  $\epsilon_i$ :

$$y_i = xw_i + \eta_i\epsilon_i. \quad (8)$$

The vector of factor loadings  $w_i$  determines the relative sensitivity of obligor  $i$  to the risk factors, and the weight  $\eta_i$  determines the relative importance of idiosyncratic risk for the obligor. The  $x$  are assumed to be normally distributed with mean zero and variance–covariance matrix  $\Omega$ .<sup>4</sup> Without loss of generality, assume there are ones on the diagonal of  $\Omega$ , so the marginal distributions are all

<sup>3</sup> The full CreditMetrics also accommodates more complex asset types, including loan commitments and derivatives contracts. See the CreditMetrics Technical Document, Chapter 4, and Finger (1998). These features are not addressed in this paper.

<sup>4</sup> In the CreditMetrics Technical Document, it is recommended that the  $x$  be taken to be stock market indexes, because the ready availability of historical data on stock indexes simplifies calibration of the covariance matrix  $\Omega$  and the weights  $w_i$ . The mathematical framework of the model, however, imposes no specific identity on the  $x$ .

$N(0, 1)$ . The  $\epsilon_i$  are assumed to be iid  $N(0, 1)$ . Again without loss of generality, it is imposed that  $y_i$  has variance 1 (i.e., that  $w_i'\Omega w_i + \eta_i^2 = 1$ ). Associated with each start-of-period rating grade  $\zeta$  is a cut-off value  $C_\zeta$ . When the latent variable  $y_i$  falls under the cut-off  $C_{\zeta(i)}$ , the obligor defaults. That is, default occurs if

$$xw_i + \eta_i\epsilon_i < C_{\zeta(i)}. \quad (9)$$

The  $C_\zeta$  values are set so that the unconditional default probability for a grade  $\zeta$  obligor is  $\bar{p}_\zeta$ , i.e., so that  $\bar{p}_\zeta = \Phi(C_\zeta)$ , where  $\Phi$  is the standard normal cdf and the  $\bar{p}_\zeta$  are defined as in Section 2.1.

The model is estimated by Monte Carlo simulation. To obtain a single trial for the portfolio, we first draw a single vector  $x$  as a multivariate  $N(0, \Omega)$  and a set of iid  $N(0, 1)$  idiosyncratic  $\epsilon$ . We form the latent  $y_i$  for each obligor, which are compared against the cut-off values  $C_{\zeta(i)}$  to determine default status  $D_i$  (one for default, zero otherwise). Portfolio loss for this trial is given by  $\sum_i D_i \lambda L_i$ . To estimate a distribution of portfolio outcomes, we repeat this process many times. The portfolio losses for each trial are sorted to form a cumulative distribution for loss. For example, if the portfolio is simulated 100,000 times, then the estimated 99.5th percentile of the loss distribution is given by the 99,500th element of the sorted loss outcomes.

### 3. Mapping between the models

Presentation of the restricted version of CreditMetrics and the use of a similar notation in outlining the models both serve to emphasize the fundamental similarities between CreditMetrics and CreditRisk<sup>+</sup>. Nonetheless, there remain substantial differences in the mathematical methods used in each, which tend to obscure comparison of the models. In the first two parts of this section, we map each model into the mathematical framework of the other. We conclude this section with a comparative analysis in which fundamental differences between the models in functional form and distributional assumptions are distinguished from differences in technique for calibration and solution.

#### 3.1. Mapping CreditMetrics to the CreditRisk<sup>+</sup> framework

To map the restricted CM2S model into the CreditRisk<sup>+</sup> framework, we need to derive the implied conditional default probability function  $p_i(x)$  used in Eq. (3).<sup>5</sup> Conditional on  $x$ , rearrangement of Eq. (9) shows that obligor  $i$

<sup>5</sup> Koyluoglu and Hickman (1998) present the conditional probability form for CreditMetrics as well, and note its utility in mapping CreditMetrics to the CreditRisk<sup>+</sup> framework.

defaults if and only if  $\epsilon_i < ((C_{\zeta(i)} - xw_i)/\eta_i)$ . Because the  $\epsilon_i$  are standard normal variates, default occurs with conditional probability

$$p_i(x) = \Phi((C_{\zeta(i)} - xw_i)/\eta_i). \quad (10)$$

The CreditRisk<sup>+</sup> methodology can now be applied in a straightforward manner. Conditional on  $x$ , default events are independent across obligors. Therefore, the conditional probability generating function for defaults,  $\mathcal{F}(z|x)$ , takes on exactly the same Poisson approximation form as in Eq. (4). To get the unconditional probability generating function  $\mathcal{F}(z)$ , we integrate out the  $x$ :

$$\mathcal{F}(z) = \int_{-\infty}^{\infty} \mathcal{F}(z|x)\phi_{\Omega}(x) dx, \quad (11)$$

where  $\phi_{\Omega}$  is the multivariate  $N(0, \Omega)$  pdf. The unconditional probability that exactly  $n$  defaults will occur in the portfolio is given by the coefficient on  $z^n$  in the Taylor series expansion of  $\mathcal{F}(z)$ :

$$\begin{aligned} \mathcal{F}(z) &= \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \exp(-\mu(x)) \frac{\mu(x)^n z^n}{n!} \phi_{\Omega}(x) dx \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left( \int_{-\infty}^{\infty} \exp(-\mu(x)) \mu(x)^n \phi_{\Omega}(x) dx \right) z^n, \end{aligned} \quad (12)$$

where  $\mu(x) \equiv \sum_i p_i(x)$ . These  $K$ -dimensional integrals are analytically intractable (even when the number of systemic factors  $K=1$ ), and in practice would be solved using Monte Carlo techniques.<sup>6</sup>

The remaining steps in CreditRisk<sup>+</sup> would follow similarly. That is, one would round loss exposures to integer multiples of a base unit  $v_0$ , apply the rule  $\mathcal{G}_i(z|x) = \mathcal{F}_i(z^{v(i)}|x)$  and multiply pgfs across obligors to get  $\mathcal{G}(z|x)$ . To get the unconditional pgf of losses, we integrate out the  $x$  as in Eq. (12). The result would be computationally unwieldy, but application of the method is conceptually straightforward.

### 3.2. Mapping CreditRisk<sup>+</sup> to the CreditMetrics framework

Translating in the opposite direction is equally straightforward. To go from CreditRisk<sup>+</sup> into the CM2S framework, we assign to obligor  $i$  a latent variable  $y_i$  defined by

<sup>6</sup> Note that the integrals differ only in  $n$ , so a single set of Monte Carlo draws for  $x$  allows successive solution of the integrals via a simple recurrence relation. This technique is fast relative to standard Gaussian quadrature.

$$y_i = \left( \sum_{k=1}^K x_k w_{ik} \right)^{-1} \epsilon_i. \quad (13)$$

The  $x_k$  and  $w_{ik}$  are the same gamma-distributed risk factors and factor loadings used in CreditRisk<sup>+</sup>. The idiosyncratic risk factors  $\epsilon_i$  are independently and identically distributed Exponential with parameter 1. Obligor  $i$  defaults if and only if  $y_i < \bar{p}_{\zeta(i)}$ . Observe that the conditional probability of default is given by

$$\begin{aligned} \Pr(y_i < \bar{p}_{\zeta(i)} | x) &= \Pr \left( \epsilon_i < \bar{p}_{\zeta(i)} \sum_{k=1}^K x_k w_{ik} | x \right) \\ &= 1 - \exp \left( - \bar{p}_{\zeta(i)} \sum_{k=1}^K x_k w_{ik} \right) \\ &\approx \bar{p}_{\zeta(i)} \sum_{k=1}^K x_k w_{ik} = p_i(x), \end{aligned} \quad (14)$$

where the second line follows using the cdf for the exponential distribution, and the last line relies on the same approximation formula as Eq. (3). The unconditional probability of default is simply  $\bar{p}_{\zeta(i)}$ , as required.

In the ordinary CreditMetrics specification, the latent variable is a linear sum of normal random variables. When CreditRisk<sup>+</sup> is mapped to the CreditMetrics framework, the latent variable takes a multiplicative form, but the idea is the same.<sup>7</sup> In CreditMetrics, the cut-off values  $C_\zeta$  are determined as functions of the associated unconditional default probabilities  $\bar{p}_\zeta$ . Here, the cut-off values are simply the  $\bar{p}_\zeta$ . Other than these differences in form, the process is identical. A single portfolio simulation trial would consist of a single random draw of sector risk factors and a single vector of random draws of idiosyncratic risk factors. From these, the obligors' latent variables are calculated, and these in turn determine default events.<sup>8</sup>

### 3.3. Essential and inessential differences between the models

CreditMetrics and CreditRisk<sup>+</sup> differ in distributional assumptions and functional forms, solution techniques, suggested methods for calibration, and mathematical language. As the preceding analysis makes clear, only the differences in distributional assumptions and functional forms are fundamental.

<sup>7</sup> One could quasi-linearize Eq. (13) by taking logs, but little would be gained because the log of the weighted sum of  $x$  variables would not simplify.

<sup>8</sup> In practice, it would be faster and more accurate to simulate outcomes as independent Bernoulli( $p_i(x)$ ) draws, which is the method used in Section 5. The latent variable method is presented only to emphasize the structural similarities between the models.

Each model can be mapped into the mathematical language of the other, which demonstrates that the difference between the latent variable representation of CreditMetrics and the covarying default probabilities of CreditRisk<sup>+</sup> is one of presentation and not substance. Similarly, methods for parameter calibration suggested in model technical documents are helpful to users, but not in any way intrinsic to the models.

By contrast, distributional assumptions and functional forms are model primitives. In each model, the choice of distribution for the systemic risk factors  $x$  and the functional form for the conditional default probabilities  $p_i(x)$  together give shape to the joint distribution over obligor defaults in the portfolio. The CreditMetrics specifications of normally distributed  $x$  and of Eq. (10) for the  $p_i(x)$  may be somewhat arbitrary, but nonetheless strongly influence the results.<sup>9</sup> One could substitute any member of the symmetric stable class of distributions (of which the normal distribution is only a special case) without requiring significant change to the CreditMetrics methods of model calibration and simulation. Even if parameters were recalibrated to yield the same mean and variance of portfolio loss, the overall shape of the loss distribution would differ, and therefore the tail percentile values would change as well. The choice of the gamma distribution and the function form for conditional default probabilities given by Eq. (1) are similarly characteristic of CreditRisk<sup>+</sup>. Indeed, in Section 6 we will show how small deviations from the gamma specification lead to significant differences in tail percentile values in a generalized CreditRisk<sup>+</sup> framework.

Remaining differences between the two models are attributable to differences in solution method. The Monte Carlo method of CreditMetrics is flexible but computationally intensive. CreditRisk<sup>+</sup> offers the efficiency of a closed-form solution, but at the expense of additional restrictions or approximations. In particular,

- CreditMetrics allows naturally for multi-state outcomes and for uncertainty in recoveries, whereas the closed-form CreditRisk<sup>+</sup> is a two-state model with fixed recovery rates.<sup>10</sup>

<sup>9</sup> Recall that Eq. (10) follows from the specification in Eq. (8) of latent variable  $y_i$  and the assumption of normally distributed idiosyncratic  $\epsilon_i$ . Therefore, this discussion incorporates those elements of the CreditMetrics model.

<sup>10</sup> In principle, both restrictions in CreditRisk<sup>+</sup> can be relaxed without resorting to a Monte Carlo methodology. It is feasible to introduce idiosyncratic risk in recoveries to CreditRisk<sup>+</sup>, but probably at the expense of the computational facility of the model. CreditRisk<sup>+</sup> also can be extended to, say, a three-state model in which the third state represents a severe downgrade (short of default). However, CreditRisk<sup>+</sup> cannot capture the exclusive nature of the outcomes (i.e., one cannot impose the mutual exclusivity of a severe downgrade and default). For this reason, as well as the Poisson approximation, the third state would need to represent a low probability event. Thus, it would be impractical to model ordinary rating migrations in CreditRisk<sup>+</sup>.

- CreditRisk<sup>+</sup> imposes a “Poisson approximation” on the conditional distribution of defaults.
- CreditRisk<sup>+</sup> rounds each obligor’s loan loss exposure to the nearest element in a finite set of values.

Using the techniques of Section 3.2, it is straightforward to construct a Monte Carlo version of CreditRisk<sup>+</sup> which avoids Poisson and loss exposure approximations and allows recovery risk. It is less straightforward but certainly possible to create a Monte Carlo multi-state generalization of CreditRisk<sup>+</sup>. Because computational convenience may be a significant advantage of CreditRisk<sup>+</sup> for some users, the effect of the Poisson and loss exposure approximations on the accuracy of CreditRisk<sup>+</sup> results will be examined in Sections 5 and 6. However, the effect of multi-state outcomes and recovery uncertainty on the distribution of credit loss will be left for future study. Therefore, our simulations will compare CreditRisk<sup>+</sup> to only the restricted CM2S version of CreditMetrics.

It is worth noting that there is no real loss of generality in the assumption of independence across sector risk factors in CreditRisk<sup>+</sup>. In each model, the vector of factor loadings ( $w$ ) is free, up to a scaling restriction. In CreditMetrics, the sector risk factors  $x$  could be orthogonalized and the correlations incorporated into the  $w$ .<sup>11</sup> However, the need to impose orthogonality in CreditRisk<sup>+</sup> does imply that greater care must be given to identifying and calibrating sectoral risks in that model.

#### 4. Calibration and main simulation results

The remaining sections of this paper study the two models using comparative simulations. The primary goal is to develop reliable intuition for how the two models will differ when applied to real world portfolios. In pursuit of this goal, we also will determine the parameters or portfolio characteristics to which each model is most sensitive. Emphasis is placed on relevance and robustness. By relevance, we mean that the simulated portfolios and calibrated parameters ought to resemble their real world counterparts closely enough for conclusions to be transferable. By robustness, we mean that the conclusions ought to be qualitatively valid over an empirically relevant range of portfolios.

This section will present our main simulations. First, in Section 4.1, we construct a set of “test deck” portfolios. All assets are assumed to be ordinary term loans. The size distribution of loans and their distribution across S&P rating grades are calibrated using data from two large samples of midsize and large corporate loans. Second, in Section 4.2, default probabilities and corre-

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<sup>11</sup> In this case, the original weights  $w$  would be replaced by  $\Omega^{1/2}w$ .

lation structures in each model are calibrated using historical default data from the S&P ratings universe. We calibrate each model to a one year risk-horizon. The main simulation results are presented in Section 4.3.

#### 4.1. Portfolio construction

Construction of our simulated loan portfolios requires choices along three dimensions. The first is *credit quality*, i.e., the portion of total dollar outstandings in each rating grade. The second is *obligor count*, i.e., the total number of obligors in the portfolio. The third is *concentration*, i.e., the distribution of dollar outstandings within a rating grade across the obligors within that grade. Note that the total portfolio dollar outstandings is immaterial, because losses will be calculated as a percentage of total outstandings.

The range of plausible credit quality is represented by four credit quality distributions, which are labelled “High”, “Average”, “Low” and “Very Low”. The first three distributions are constructed using data from internal Federal Reserve Board surveys of large banking organizations.<sup>12</sup> The “Average” distribution is the average distribution across the surveyed banks of total outstandings in each S&P grade. The “High” and “Low” distributions are drawn from the higher and lower quality distributions found among the banks in the sample. The “Very Low” distribution is not found in the Federal Reserve sample, but is intended to represent a very weak large bank loan portfolio during a recession. Speculative grade (BB and below) loans account for half of outstandings in the “Average” portfolio, and 25%, 78% and 83% in the “High”, “Low”, “Very Low” quality portfolios, respectively. The distributions are depicted in Fig. 1.

Realistic calibration of obligor count is likely to depend not only on the size of the hypothetical bank, but also on the bank’s business focus. A very large bank with a strong middle-market business might have tens of thousands of rated obligors in its commercial portfolio. A bank of the same size specializing in the large corporate market might have only a few thousand. For the “base case” calibration, we set  $N = 5000$ . To establish robustness of the conclusions to the choice of  $N$ , we model portfolios of 1000 and 10,000 obligors as well. In all simulations, we assume each obligor is associated with only one loan in the portfolio.

Portfolio concentration is calibrated in two stages. First, we divide the total number of obligors  $N$  across the rating grades. Second, for each rating grade, we determine how the total exposure within the grade is distributed across the number of obligors in the grade. In both stages, distributions are calibrated

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<sup>12</sup> Each bank provided the amount outstanding, by rating grade, in its commercial loan book.

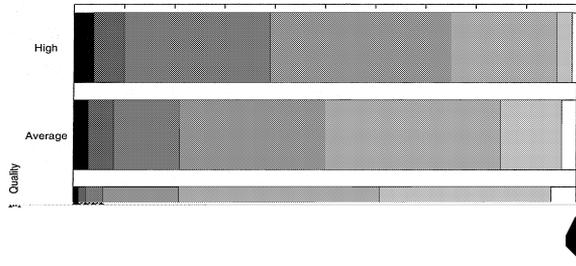


Fig. 1. Credit quality distributions.

using the Society of Actuaries (1996, hereafter cited as “SoA”) sample of mid-sized and large private placement loans (see also Carey, 1998).

Let  $q_\zeta$  be the dollar volume of exposure to rating grade  $\zeta$  as a share of total portfolio exposure (determined by the chosen “credit quality distribution”), and let  $v_\zeta$  be the mean book value of loans in rating grade  $\zeta$  in the SoA data. We determine  $n_\zeta$ , the number of obligors in rating grade  $\zeta$ , by imposing

$$q_\zeta = n_\zeta v_\zeta / \sum_g n_g v_g \tag{15}$$

for all  $\zeta$ . That is, the  $n_\zeta$  are chosen so that, in a portfolio with mean loan sizes in each grade matching the SoA mean sizes, the exposure share of that grade matches the desired share  $q_\zeta$ . The equations of form (15) are easily transformed into a set of six linearly independent equations and seven unknown  $n_\zeta$  values (using the S&P eight grade scale). Given the restriction  $\sum n_\zeta = N$ , the vector  $n$  is uniquely determined.<sup>13</sup> Table 1 shows the values of the vector  $n$  associated with each credit quality distribution when  $N = 5000$ .

The final step is to distribute the  $q_\zeta$  share of total loan exposure across the  $n_\zeta$  obligors in each grade. For our “base case” calibration, the distribution within grade  $\zeta$  is chosen to match (up to a scaling factor) the distribution for grade  $\zeta$  exposures in the SoA data. The SoA loans in grade  $\zeta$  are sorted from smallest to largest, and used to form a cumulative distribution  $H_\zeta$ . The size of the  $j$ th exposure,  $j = 1, \dots, n_\zeta$ , is set to the  $(j - 1/2)/n_\zeta$  percentile of  $H_\zeta$ .<sup>14</sup> Finally, the

<sup>13</sup> After solving the system of linear equations, the values of  $n_\zeta$  are rounded to whole numbers.

<sup>14</sup> For example, say  $n_\zeta = 200$  and there are 6143 loans in the grade  $\zeta$  SoA sample. To set the size of, say, the seventh ( $j = 7$ ) simulated loan, we calculate the index  $(j - 1/2)6143/n_\zeta \approx 199.65$ . The loan size is then formed as an interpolated value between the 199th and 200th loans in the SoA vector.

Table 1  
Number of obligors in each rating grade

	Portfolio credit quality			
	High	Average	Low	Very Low
AAA	191	146	50	25
AA	295	250	77	51
A	1463	669	185	158
BBB	1896	1558	827	660
BB	954	1622	1903	1780
B	136	556	1618	1851
CCC	65	199	340	475
Total	5000	5000	5000	5000

$n_\zeta$  loan sizes are normalized to sum to  $q_\zeta$ . This method ensures that the shape of the distribution of loan sizes will not be sensitive to the choice of  $n_\zeta$  (unless  $n_\zeta$  is very small). As an alternative to the base case, we also model a portfolio in which all loans within a rating grade are equal-sized, i.e., each loan in grade  $\zeta$  is of size  $q_\zeta/n_\zeta$ .

In all simulations below, it is assumed that loss given default is a fixed proportion  $\lambda = 0.3$  of book value, which is consistent with historical loss given default experience for senior unsecured bank loans.<sup>15</sup> Percentile values on the simulated loss distributions are directly proportional to  $\lambda$ . Holding fixed all other model parameters and the permitted probability of bank insolvency, the required capital given a loss rate of, say,  $\lambda = 0.45$  would simply be 1.5 times the required capital given  $\lambda = 0.3$ .

It should be emphasized that the SoA data are used to impose *shape*, but not *scale*, on the distributions of loan sizes. The ratio of the mean loan size in grade  $\zeta_1$  to the mean loan size in grade  $\zeta_2$  is determined by the corresponding ratio in the SoA sample. Within each grade, SoA data determine the ratio of any two percentile values of loan size (e.g., the 75th percentile to the median). However, measures of portfolio concentration (e.g., the ratio of the sum of the largest  $j$  loans to the total portfolio value) depend strongly on the choice of  $N$ , and thus not only on SoA sample.

Finally, CreditRisk<sup>+</sup> requires a discretization of the distribution of exposures, i.e., the selection of the base unit of loss  $v_0$ . In the main set of simulations, we will set  $v_0$  to  $\lambda$  times the fifth percentile value of the distribution of loan sizes. In Section 5, it will be shown that simulation results are quite robust to the choice of  $v_0$ .

<sup>15</sup> See the CreditMetrics Technical Document, Section 7.1.2.

#### 4.2. Default probabilities and correlations

In any model of portfolio credit risk, the structure of default rate correlations is an important determinant of the distribution of losses. Special attention must therefore be given to mutually consistent calibration of parameters which determine default correlations. In the exercises below, we calibrate CreditMetrics and CreditRisk<sup>+</sup> to yield the same unconditional expected default rate for an obligor of a given rating grade, and the same default correlation between any two obligors within a single rating grade.<sup>16</sup>

For simplicity, we assume a single systemic risk factor  $x$ .<sup>17</sup> Within each rating grade, obligors are statistically identical (except for loan size). That is, every obligor of grade  $\zeta$  has unconditional default probability  $\bar{p}_\zeta$  and has the same weight  $w_\zeta$  on the systemic risk factor. (The value of  $w_\zeta$  will, of course, depend on the choice of model.) The  $\bar{p}$  values are set to the long-term average annual default probabilities given in Table 6.9 of the CreditMetrics Technical Document, and are shown below in the first column of Table 2. For a portfolio of loans, this is likely to be a relatively conservative calibration of mean annual default probabilities.<sup>18</sup>

The weights  $w_\zeta$  are calibrated for each model by working backwards from the historical volatility of annual default rates in each rating grade. First, using data in Brand and Bahar (1998, Table 12) on historical default experience in each grade, we estimate the variance  $V_\zeta$  of the conditional default rate  $p_\zeta(x)$ . The estimation method and results are described in Appendix B. For calibration purposes, the default rate volatilities are most conveniently expressed as normalized standard deviations  $\sqrt{V_\zeta}/\bar{p}_\zeta$ . The values assumed in the simulations are shown in the second column of Table 2, and the implied default correlations  $\rho_\zeta$  for any two obligors in the same rating grade are shown in the third column. To confirm the qualitative robustness of the results, additional simulations will be presented in Section 5 in which the assumed normalized volatilities are twice the values used here.

The second step in determining the  $w_\zeta$  is model-dependent. To calibrate the CreditMetrics weights, we use Proposition 1:

**Proposition 1.** *In the CreditMetrics model,*

$$V_\zeta \equiv \text{Var}[p_\zeta(x)] = \Phi(C_\zeta, C_\zeta, w_\zeta^2) - \bar{p}_\zeta^2, \quad (16)$$

<sup>16</sup> Koyluoglu and Hickman (1998) also use these two moments to harmonize calibration of somewhat more restrictive versions of the two models.

<sup>17</sup> This is in the same spirit as the “Z-risk” approach of Belkin et al. (1998).

<sup>18</sup> Carey (1998) observes that default rates on speculative grade private placement loans tend to be lower than on publicly held bonds of the same senior unsecured rating, and attributes this superior performance to closer monitoring.

Table 2  
Default rate volatility and factor weights<sup>a</sup>

	Historical experience			Factor loadings			
	$\bar{p}$	$\sqrt{V/\bar{p}}$	$\rho$	CM2S	CR <sup>+</sup>	CR <sup>+</sup>	CR <sup>+</sup>
$\sigma$					1.0	1.5	4.0
AAA	0.01	1.4	0.0002	0.272	1.400	0.933	0.350
AA	0.02	1.4	0.0004	0.285	1.400	0.933	0.350
A	0.06	1.2	0.0009	0.279	1.200	0.800	0.300
BBB	0.18	0.4	0.0003	0.121	0.400	0.267	0.100
BB	1.06	1.1	0.0130	0.354	1.100	0.733	0.275
B	4.94	0.55	0.0157	0.255	0.550	0.367	0.138
CCC	19.14	0.4	0.0379	0.277	0.400	0.267	0.100

<sup>a</sup> Unconditional annual default probabilities  $\bar{p}$  taken from the CreditMetrics Technical Document, Table 6.9, and are expressed here in percentage points. Historical experience for default rate volatility derived from Brand and Bahar (1998, Table 12), as described below in Appendix B.

where  $\Phi(z_1, z_2, \rho)$  is the bivariate normal cdf for  $Z \equiv [Z_1 Z_2]'$  such that

$$E[Z] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{Var}[Z] = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}. \quad (17)$$

The proof is given in Appendix C. Given the cut-off values  $C_\zeta$  (which are functions of the  $\bar{p}_\zeta$ ) and normalized volatilities  $\sqrt{V_\zeta/\bar{p}_\zeta}$ , non-negative  $w_\zeta$  are uniquely determined by the non-linear Eq. (16). The solutions are shown in the fourth column of Table 2.

In CreditRisk<sup>+</sup>, a model with a single systemic risk factor and obligor-specific idiosyncratic risk can be parameterized flexibly as a two risk factor model in which the first risk factor has zero volatility and thus always equals one.<sup>19</sup> Let  $w_\zeta$  be the weights on  $x_2$  (which are constant across obligors within a grade but allowed to vary across grades), so the weights on  $x_1$  are  $1 - w_\zeta$ . To simplify notation, we set the first risk factor ( $x_1$ ) identically equal to one, and denote the second risk factor ( $x_2$ ) as  $x$  and the standard deviation  $\sigma_2$  of  $x_2$  as  $\sigma$ . Under this specification, the variance of the default probability in CreditRisk<sup>+</sup> for a grade  $\zeta$  obligor is

$$V_\zeta = \text{Var}[\bar{p}_\zeta(1 - w_\zeta + w_\zeta x)] = (\bar{p}_\zeta w_\zeta \sigma)^2 \quad (18)$$

so the normalized volatility  $\sqrt{V_\zeta/\bar{p}_\zeta}$  equals  $w_\zeta \sigma$ .

Given  $\sigma$ , the weights  $w_\zeta$  are uniquely determined. However, there is no obvious additional information to bring to the choice of  $\sigma$ . This might appear

<sup>19</sup> See the CreditRisk<sup>+</sup> manual, Section A12.3. The first factor is referred to as a “specific factor”. Because it represents diversifiable risk, it contributes no volatility to a well diversified portfolio.

to make little or no difference, because the volatility of the default probabilities depends only on the product  $w_\zeta\sigma$ . However, because  $\sigma$  controls the shape (and not merely the scale) of the distribution of  $x$ , higher moments of the distribution of  $p_i(x)$  depend directly on  $\sigma$  and not only on the product ( $w_\zeta\sigma$ ). Consequently, tail probabilities for portfolio loss are quite sensitive to the choice of  $\sigma$ . To illustrate this sensitivity, simulation results will be presented for three values of  $\sigma$  (1.0, 1.5, and 4.0). See the last three columns of Table 2 for the values of  $w_\zeta$  corresponding to each of these  $\sigma$  (with  $w_\zeta\sigma$  held constant) and the grade-specific normalized volatilities.

In the CreditRisk<sup>+</sup> manual, Section A7.3, it is suggested that  $\sigma$  is roughly one. This estimate is based on a single-sector calibration of the model, which is equivalent to setting all the  $w_\zeta$  to one. The exposure-weighted average of values in the second column of Table 2 would then be a reasonable calibration of  $\sigma$ . For most portfolios, this would yield  $\sigma \approx 1$ , as suggested. Our specification is strictly more general than the single-sector approach, because it allows the relative importance of systemic risk to vary across rating grades, and also is more directly comparable to our calibrated correlation structure for CreditMetrics. Note that the difficulty of calibrating  $\sigma$  in this more general specification should not be interpreted as a disadvantage to CreditRisk<sup>+</sup> relative to CreditMetrics, because CreditMetrics avoids this calibration issue by fiat. In assuming the normal distribution for the systemic risk factor, CreditMetrics is indeed imposing very strong restrictions on the shape of the distribution tail.

When  $\sigma = 1$  is used in our calibration of CreditRisk<sup>+</sup>, a problem arises in that some of the factor loadings exceed one. Such values imply negative weights on the specific factors, which violate both intuition and the formal assumptions of the model. However, CreditRisk<sup>+</sup> can tolerate negative weights so long as all coefficients in the polynomial expansion of the portfolio loss probability generating function remain positive<sup>20</sup>. For the weights in the  $\sigma = 1$  column, we have confirmed numerically that our simulations always produce valid loss distributions.<sup>21</sup>

To users of CreditRisk<sup>+</sup>, this “top-down” approach to calibration should seem entirely natural. Users of CreditMetrics, however, may find it some what

<sup>20</sup> Conditional on small realizations of  $x$ , an obligor with negative weight on the specific factor can have a negative default probability. However, so long as such obligors are relatively few and their negative weights relatively small in magnitude, the portfolio loss distribution can still be well-behaved. There may be some similarity to the problem of generating default probabilities *over one* conditional on large realizations of  $x$ , which need not cause any problem at the portfolio level, so long as the portfolio does not have too many low-rated obligors with high loading on the systemic risk factor.

<sup>21</sup> The weights in the  $\sigma = 1.5$  and  $\sigma = 4$  columns of Table 2 are all bounded in  $(0, 1)$  so the issue does not arise.

alien. The design of CreditMetrics facilitates a detailed specification of risk factors (e.g., to represent industry- and country-specific risks), and thereby encourages a “bottom-up” style of calibration. From a mathematical point-of-view, top-down calibration of CreditMetrics is equally valid, and is convenient for purposes of comparison with CreditRisk<sup>+</sup>. As an empirical matter, a top-down approach ought to work as well as a bottom-up approach on a broadly diversified bond portfolio, because the top-down within-grade default correlation should roughly equal the average of the bottom-up default correlations among obligors within that grade. The bottom-up approach, however, is better suited to portfolios with industry or geographic concentrations.

Another limitation of our method of calibration is that it makes no use of historical default correlations between obligors of different grades. Cross-grade default correlations are determined as artifacts of the models’ functional forms for  $p_{\zeta}(x)$ , the assumption of a single systemic risk factor, and the chosen factor loadings. In general, our calibrations for CreditMetrics and CreditRisk<sup>+</sup> yield quite similar cross-grade default correlations, but there are discrepancies for some cells. For example, the default correlation between a BB issuer and a CCC issuer is 0.0204 in CreditMetrics and 0.0222 in CreditRisk<sup>+</sup>. Thus, while our method equalizes variance of loss across the two models given homogeneous (i.e., single grade) portfolios, there will be slight differences given mixed-grade portfolios.

#### 4.3. Main simulation results

Results for the main set of simulations are displayed in Table 3.<sup>22</sup> Each quadrant of the table shows summary statistics and selected percentile values for CreditMetrics and CreditRisk<sup>+</sup> portfolio loss distributions for a portfolio of a given credit quality distribution. The summary statistics are the mean, standard deviation, index of skewness and index of kurtosis. The latter two are defined for a random variable  $y$  by

$$\text{Skewness}(y) = \frac{E[(y - E[y])^3]}{\text{Var}[y]^{3/2}}, \quad \text{Kurtosis}(y) = \frac{E[(y - E[y])^4]}{\text{Var}[y]^2}.$$

Skewness is a measure of the asymmetry of a distribution, and kurtosis is a measure of the relative thickness of the tails of the distribution. For portfolio

<sup>22</sup> In these simulations, there are  $N=5000$  loans in the portfolio, grade-specific loan size distributions are taken from the SoA sample, average severity of loss is held constant at 30%, and the weights  $w_{\zeta}$  and CreditRisk<sup>+</sup> parameter  $\sigma$  are taken from Table 2. CreditMetrics distributions are formed using 200,000 portfolio trials.

credit risk models, high kurtosis indicates a relatively high probability of very large credit losses. These summary statistics are calculated analytically for the CreditRisk<sup>+</sup> model using the results of Gordy (1999), and are approximated for CreditMetrics from the Monte Carlo loss distribution.

The percentile values presented in the table are the loss levels associated with the 50% (median), 75%, 95%, 99%, 99.5% and 99.97% points on the cumulative distribution of portfolio losses. In many discussions of credit risk modeling, the 99th and sometimes the 95th percentiles of the distribution are taken as points of special interest. The 99.5th and 99.97th percentiles may appear to be extreme tail values, but are in fact of greater practical interest than the 99th percentile. To merit an AA rating, an institution must have a probability of default over a one year horizon of roughly three basis points (0.03%).<sup>23</sup> Such an institution therefore ought to hold capital (or reserves) against credit loss equal to the 99.97th percentile value. Capitalization sufficient to absorb up to the 99.5th percentile value of losses would be consistent with only a BBB- rating.

Table 3, for the Average quality portfolio, illustrates the qualitative characteristics of the main results. The expected loss under either model is roughly 48 basis points of the portfolio book value.<sup>24</sup> The standard deviation of loss is roughly 32 basis points. When the CreditRisk<sup>+</sup> parameter  $\sigma$  is set to 1, the two models predict roughly similar loss distributions overall. The 99.5th and 99.97th percentile values are roughly 1.8% and 2.7% of portfolio book value in each case. As  $\sigma$  increases, however, the CreditRisk<sup>+</sup> distribution becomes increasingly kurtotic. The standard deviation of loss remains the same, but tail percentile values increase substantially. The 99.5th and 99.97th CreditRisk<sup>+</sup> percentile values given  $\sigma = 4.0$  are respectively 40% and 90% larger than the corresponding CreditMetrics values.

High, Low, and Very Low quality portfolios produce different expected losses (19, 93, and 111 basis points, respectively), but similar overall conclusions regarding our comparison of the two models. CreditRisk<sup>+</sup> with  $\sigma = 1.0$  produces distributions roughly similar to those of CreditMetrics, although as credit quality deteriorates the extreme percentile values in CreditRisk<sup>+</sup> increase more quickly than in CreditMetrics. As  $\sigma$  increases, so do the extreme loss percentiles.

Overall, capital requirements implied by these simulations may seem relatively low. Even with a Low quality portfolio, a bank would need to hold only 4.5–6% capital against credit risk in order to maintain an AA

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<sup>23</sup> This is a rule of thumb often used by practioners. Following the CreditMetrics Technical Document, we have taken a slightly lower value (0.02%) as the AA default probability.

<sup>24</sup> For this credit quality distribution, the expected annual default rate is 1.6% (by loan value). Multiply by the average severity of 30% to get a loss of 48 basis points.

Table 3  
CreditMetrics vs CreditRisk<sup>+</sup>: Main simulations

$\sigma$	High quality portfolio				Average quality portfolio			
	CM2S	CR <sup>+</sup>			CM2S	CR <sup>+</sup>		
		1.00	1.50	4.00		1.00	1.50	4.00
Mean	0.194	0.194	0.194	0.194	0.481	0.480	0.480	0.480
Std Dev	0.152	0.156	0.156	0.156	0.319	0.325	0.325	0.325
Skewness	1.959	1.874	2.537	5.848	1.696	1.854	2.633	6.527
Kurtosis	9.743	8.432	13.531	65.004	8.137	8.374	14.220	75.694
0.5000	0.156	0.150	0.148	0.160	0.409	0.391	0.384	0.414
0.7500	0.257	0.257	0.240	0.222	0.624	0.612	0.567	0.520
0.9500	0.486	0.501	0.497	0.398	1.089	1.120	1.116	0.869
0.9900	0.733	0.745	0.794	0.858	1.578	1.628	1.749	1.916
0.9950	0.847	0.850	0.928	1.121	1.795	1.847	2.033	2.488
0.9997	1.342	1.277	1.490	2.345	2.714	2.736	3.225	5.149
$\sigma$	Low quality portfolio				Very Low quality portfolio			
	CM2S	CR <sup>+</sup>			CM2S	CR <sup>+</sup>		
		1.00	1.50	4.00		1.00	1.50	4.00
Mean	0.927	0.927	0.927	0.927	1.107	1.106	1.106	1.106
Std Dev	0.557	0.566	0.566	0.566	0.635	0.644	0.644	0.644
Skewness	1.486	1.883	2.734	6.990	1.393	1.885	2.747	7.060
Kurtosis	6.771	8.511	14.873	82.988	6.299	8.523	14.961	84.097
0.5000	0.809	0.769	0.753	0.815	0.977	0.926	0.906	0.979
0.7500	1.194	1.154	1.063	0.967	1.418	1.364	1.259	1.146
0.9500	1.989	2.045	2.041	1.585	2.316	2.379	2.376	1.854
0.9900	2.782	2.936	3.161	3.481	3.187	3.395	3.654	4.024
0.9950	3.124	3.320	3.664	4.504	3.562	3.832	4.227	5.192
0.9997	4.558	4.877	5.770	9.251	5.105	5.607	6.631	10.618

rating standard.<sup>25</sup> It should be noted, however, that these simulations assume uniform default correlations within each rating grade. In real world portfolios, there may sometimes be pockets of higher default correlation, due perhaps to imperfect geographic or industry diversification. Furthermore, it should be emphasized that these simulations incorporate only default risk, and thus additional capital must be held for other forms of risk, including market risk, operational risk, and recovery uncertainty.

<sup>25</sup> Simulations by Carey (1998) suggest somewhat higher capital requirements. His simulations account for recovery risk, which is assumed away here. Perhaps more importantly, his simulations are calibrated using data from 1986 to 1992, which was a relatively unfavorable period in the credit cycle.

## 5. Robustness of model results

In this section, we explore the sensitivity of the models to parameter calibration and portfolio construction.

*Obligor count:* Compared to portfolios of equities, loan portfolios can be quite large and still receive substantial diversification benefits from adding more obligors.<sup>26</sup> Table 4 compares CreditMetrics and CreditRisk<sup>+</sup> results for Average quality portfolios of 1000, 5000, and 10,000 obligors. Even with portfolios of this size, increasing the number of obligors reduces risk significantly. The standard deviation of the 10,000 obligor portfolio is roughly 20% less than that of the 1000 obligor portfolio, and the 99.5th and 99.97th percentile values fall by 13–15%. However, the qualitative nature of the results, particularly the comparison between the two models, remains unchanged.

*Loan size distribution:* The loan size distributions derived from the SoA data are likely to be somewhat skew in comparison with real bank portfolios. For  $N=5000$ , the largest loans are over 0.65% of the portfolio, which is not much below supervisory concentration limits. To examine the effect of loan size distribution, we construct portfolios in which all loans within a single rating grade have the same size. Results are shown in Table 5 for Average quality portfolios. The tail percentiles are somewhat lower for the equal-sized portfolios, but, if one considers the magnitude of difference between the two loan-size distributions, the difference in model outputs seems minor. These results suggest that, with real bank portfolios, neither model is especially sensitive to the distribution of loan sizes.

*Normalized volatilities:* Due to the empirical difficulty of estimating default correlations with precision, practitioners may be especially concerned with the sensitivity of the results to the values of the normalized volatilities in Table 2. Therefore, we calibrate and run a set of simulations in which normalized volatilities are double the values used above. CreditMetrics weights  $w_{\zeta}$  increase substantially, though not quite proportionately.<sup>27</sup> We retain the same CreditRisk<sup>+</sup>  $w_{\zeta}$  values given in the last three columns of Table 2, but double the respective  $\sigma$  values.

Results are presented in Table 6 for the Average quality portfolio. As should be expected, extreme tail percentile values increase substantially. Compared to the values in Table 3, the 99.97th percentile values nearly triple. Similar increases in tail percentile values are observed for the other credit quality distributions.

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<sup>26</sup> Essentially, this is because risk in loans is dominated by large changes in value which occur with relatively low probability. The skew distribution of individual losses allow the tail of the portfolio loss distribution to thin with diversification at only a relatively slow rate.

<sup>27</sup> Due to the nonlinearity of the normal cdf, a given percentage increase in the normalized volatility is generally associated with a somewhat smaller percentage increase in the weight on  $x$ .

Table 4  
Effect of obligor count on portfolio loss distributions<sup>a</sup>

	$N = 1000$		$N = 5000$		$N = 10,000$	
	CM2S	CR <sup>+</sup>	CM2S	CR <sup>+</sup>	CM2S	CR <sup>+</sup>
Mean	0.480	0.480	0.481	0.480	0.480	0.480
Std Dev	0.387	0.398	0.319	0.325	0.306	0.312
Skewness	1.672	2.245	1.696	2.633	1.734	2.788
Kurtosis	7.442	11.434	8.137	14.220	8.390	15.202
0.5000	0.383	0.370	0.409	0.384	0.410	0.380
0.7500	0.653	0.619	0.624	0.567	0.615	0.552
0.9500	1.235	1.251	1.089	1.116	1.064	1.097
0.9900	1.803	1.957	1.578	1.749	1.531	1.719
0.9950	2.044	2.278	1.795	2.033	1.750	1.999
0.9997	3.093	3.626	2.714	3.225	2.653	3.169

<sup>a</sup> Average quality portfolio with SoA loan size distributions. All CreditRisk<sup>+</sup> simulations use  $\sigma = 1.5$ .

Table 5  
Equal-sized vs SoA loan sizes<sup>a</sup>

	SoA loan sizes		Equal-sized loans	
	CM2S	CR <sup>+</sup>	CM2S	CR <sup>+</sup>
Mean	0.481	0.480	0.481	0.480
Std Dev	0.319	0.325	0.299	0.303
Skewness	1.696	2.633	1.801	2.924
Kurtosis	8.137	14.220	8.712	16.047
0.5000	0.409	0.384	0.412	0.373
0.7500	0.624	0.567	0.609	0.549
0.9500	1.089	1.116	1.051	1.101
0.9900	1.578	1.749	1.527	1.728
0.9950	1.795	2.033	1.747	2.009
0.9997	2.714	3.225	2.649	3.187

<sup>a</sup> Average quality portfolio of  $N = 5000$  obligors. CreditRisk<sup>+</sup> simulations use  $\sigma = 1.5$ .

*Discretization of loan sizes:* In the main simulations, the CreditRisk<sup>+</sup> base exposure unit is set to  $\lambda$  times the fifth percentile value of the distribution of loan sizes. At least locally, the error introduced by this discretization is negligible. We have run most of our simulations with  $v_0$  set to  $\lambda$  times the 2.5th and 10th percentile values. For both these alternatives, the percentile values of the loss distribution differed from those of the main simulations by no more than 0.0005.

## 6. Modified CreditRisk<sup>+</sup> specifications

The analysis of Section 4.3 demonstrates the sensitivity of CreditRisk<sup>+</sup> to the calibration of  $\sigma$ . When we vary  $\sigma$  while holding the  $w_\tau\sigma$  constant, the

Table 6  
Effect of increased default volatilities<sup>a</sup>

$\sigma$	CM2S	CR <sup>+</sup>		
		2.00	3.00	8.00
Mean	0.480	0.480	0.480	0.480
Std Dev	0.590	0.611	0.611	0.611
Skewness	3.221	3.903	5.774	15.131
Kurtosis	20.278	26.209	54.267	359.274
0.5000	0.287	0.265	0.313	0.400
0.7500	0.615	0.507	0.447	0.492
0.9500	1.597	1.648	1.442	0.710
0.9900	2.845	3.130	3.311	2.533
0.9950	3.467	3.818	4.239	4.202
0.9997	6.204	6.772	8.386	13.459

<sup>a</sup> Average quality portfolio with SoA loan size distributions.  $N = 5000$  obligors.

mean and standard deviation of loss remain unchanged, but the tail percentile values change markedly. This sensitivity is both a direct and an indirect consequence of a property of the gamma distribution for  $x$ . Unlike the normal distribution, which has kurtosis equal to 3 regardless of its variance, the kurtosis of a gamma-distributed variable depends on its parameters. A gamma random variable with mean one and variance  $\sigma^2$  has kurtosis  $3(1 + 2\sigma^2)$ , so higher  $\sigma$  imposes a more fat-tailed shape on the distribution, which is transmitted to the shape of the distribution for  $p_i(x)$  for each obligor  $i$ . So long as  $w_{\zeta(i)}\sigma$  is held constant, varying  $\sigma$  has no effect on the mean or standard deviation of  $p_i(x)$ . However, it is straightforward to show that the kurtosis of  $p_i(x)$  equals the kurtosis of  $x$ , so increasing  $\sigma$  increases the kurtosis of  $p_i(x)$ .

Increasing the kurtosis of the  $p_i(x)$  has the direct effect of increasing the thickness of the tail of the distribution for loss. This is explored below by substituting an alternative distribution for  $x$  which has mean one and variance  $\sigma^2$  but is less kurtotic. The indirect effect of higher kurtosis for  $p_i(x)$  is that it magnifies the error induced by the Poisson approximation. To explore the effect of  $\sigma$  on the size of the approximation error, we use the methods of Section 3.2 to eliminate the Poisson approximation from the calculations.

The Poisson approximation necessarily contributes to the thickness of the tail in CreditRisk<sup>+</sup> loss distributions. In a Monte Carlo based model, such as CreditMetrics, an obligor can default no more than once, so no more than  $N$  defaults can be suffered. Under the Poisson approximation, a single obligor can be counted in default any number of times (albeit with very small probabilities of multiple defaults). Thus, CreditRisk<sup>+</sup> assigns a positive probability to the number of defaults exceeding the number of obligors. No matter how the portfolio is constructed and how the two models are calibrated, there must be a

crossing point beyond which CreditRisk<sup>+</sup> percentile values all exceed the corresponding CreditMetrics percentile values.

Depending on the portfolio and the model parameters, the effect of the Poisson approximation may or may not be negligible. To test the empirical relevance of this effect, we compare CreditRisk<sup>+</sup> results against those of a Monte Carlo version of CreditRisk<sup>+</sup>. The Monte Carlo version is similar to that outlined in Section 3.2, except that default of obligor  $i$ , conditional on  $x$ , is drawn as a Bernoulli random variate with probability  $p_i(x)$ , rather than using the latent variable approach of Eq. (13). This avoids the small approximation error induced by Eq. (14), but otherwise imposes exactly the same distributional assumptions and functional forms as the standard CreditRisk<sup>+</sup> model.

We conduct similar Monte Carlo exercises to explore alternative distributional assumptions for  $x$ . Say that  $x$  is distributed such that  $x^2 \sim \text{Gamma}(\alpha, \beta)$ . As described in Appendix D, it is straightforward to solve for parameters  $(\alpha, \beta)$  such that the  $E[x] = 1$  and  $\text{Var}[x] = \sigma^2$ . Although this  $x$  matches the mean and variance of the standard CreditRisk<sup>+</sup> gamma-distributed risk factor, it is much less kurtotic. The “gamma-squared” distribution is compared to the ordinary gamma distribution in Fig. 2. The top panel plots the cdfs for a gamma distributed variable (solid line) and a gamma-squared distributed variable (dashed line). Both variables have mean one and variance one. The two distributions appear to be quite similar, and indeed would be difficult to distinguish empirically. Nonetheless, as shown in the bottom panel, the two distributions differ substantially in the tails. The 99.9th percentiles are 6.91 and 5.58 for the gamma and gamma-squared distributions, respectively. The 99.97th percentiles are 8.11 vs. 6.20.

The results of both exercises on an Average quality portfolio are shown in Table 7.<sup>28</sup> The standard CreditRisk<sup>+</sup> results for  $\sigma = 1.5$  and  $\sigma = 4.0$  (columns 1 and 4) are taken from Table 3. Results for the Monte Carlo version of CreditRisk<sup>+</sup> are shown in columns 2 and 5. For the moderate value of  $\sigma = 1.5$ , the 99.97th percentile value is reduced by under two percent. For the larger value  $\sigma = 4.0$ , however, the 99.97th percentile value is reduced by over eight percent. The higher the value of  $\sigma$ , the higher the probability of large conditional default probabilities. As the validity of the Poisson approximation thus breaks down for high  $\sigma$ , so does the accuracy of the analytic CreditRisk<sup>+</sup> methodology.

Results for the modified CreditRisk<sup>+</sup> with  $x^2$  gamma-distributed are shown in columns 3 and 6. For both values of  $\sigma$ , the mean and standard deviation of portfolio loss are roughly as before, but the tail percentiles are quite signifi-

<sup>28</sup> Qualitatively similar results are found for portfolios based on the other credit quality distributions.

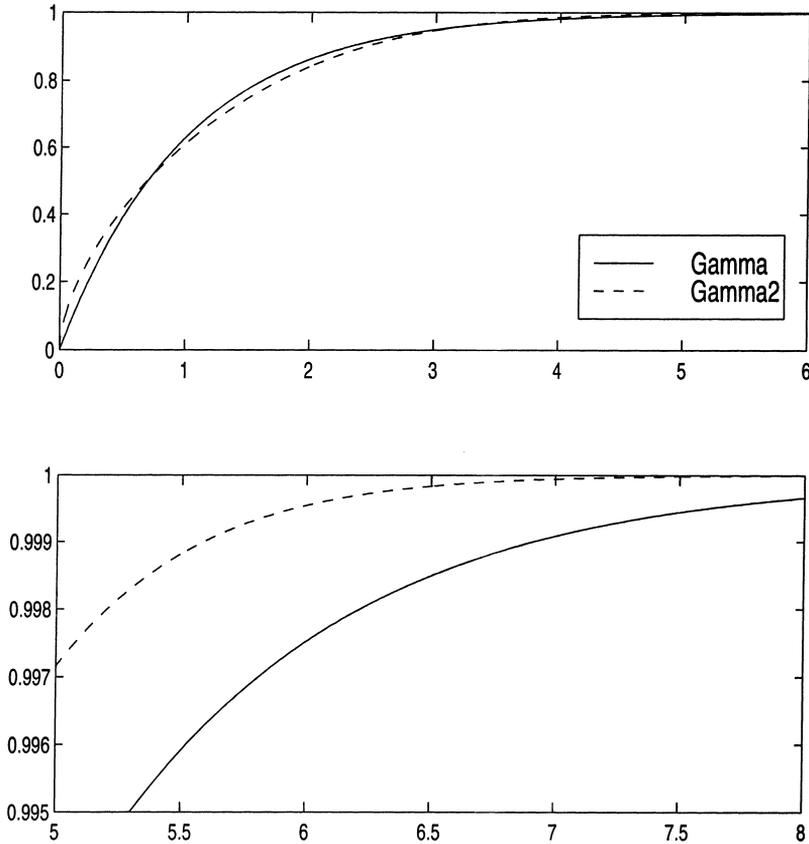


Fig. 2. Gamma and gamma-squared distributions.

*Note:* The two lines are cdfs of variables with mean one and variance one. If  $x$  is gamma distributed, the solid line is its cdf. If  $x^2$  is gamma distributed, the dashed line is its cdf.

cantly reduced. Indeed, the 99.97th percentile value for the modified model under  $\sigma = 1.5$  is even less than the corresponding CreditMetrics value. This demonstrates the critical importance of the shape of the distribution of the systemic risk factor.

## 7. Discussion

This paper demonstrates that there is no unbridgeable difference in the views of portfolio credit risk embodied in CreditMetrics and CreditRisk<sup>+</sup>. If we consider the restricted form of CreditMetrics used in the analysis, then each model can be mapped into the mathematical framework of the other, so that

Table 7  
Modified CreditRisk<sup>+</sup> models<sup>a,b</sup>

	$\sigma = 1.50$			$\sigma = 4.00$		
	CR <sup>+</sup>	CR <sup>+</sup> (MC)	CR <sup>+</sup> (X2)	CR <sup>+</sup>	CR <sup>+</sup> (MC)	CR <sup>+</sup> (X2)
Mean	0.480	0.479	0.480	0.480	0.480	0.479
Std Dev	0.325	0.319	0.322	0.325	0.316	0.321
Skewness	2.633	2.606	2.037	6.527	6.070	5.142
Kurtosis	14.220	13.938	8.342	75.694	62.525	40.076
0.5000	0.384	0.385	0.374	0.414	0.416	0.411
0.7500	0.567	0.564	0.575	0.520	0.518	0.512
0.9500	1.116	1.106	1.158	0.869	0.866	0.888
0.9900	1.749	1.719	1.670	1.916	1.904	2.047
0.9950	2.033	1.991	1.868	2.488	2.494	2.553
0.9997	3.225	3.179	2.561	5.149	4.729	4.099

<sup>a</sup> Average quality portfolio with SoA loan size distribution.  $N = 5000$  obligors. 200,000 portfolio trials in the Monte Carlo simulations.

<sup>b</sup> CR<sup>+</sup> (columns 1 and 4) is standard CreditRisk<sup>+</sup>. CR<sup>+</sup>(MC) (columns 2 and 5) is CreditRisk<sup>+</sup> estimated by Monte Carlo. CR<sup>+</sup>(X2) (columns 3 and 6) is a Monte Carlo CreditRisk<sup>+</sup> with  $\chi^2$  gamma-distributed.

the primary sources of discrepancy in results are differences in distributional assumptions and functional forms.

Simulations are constructed for a wide range of plausible loan portfolios and correlation parameters. The results suggest a number of general conclusions. First, the two models perform very similarly on an average quality commercial loan portfolio when the CreditRisk<sup>+</sup> volatility parameter  $\sigma$  is given a low value. Both models demand higher capital on lower quality portfolios, but CreditRisk<sup>+</sup> is somewhat more sensitive to credit quality than the two-state version of CreditMetrics. It should be emphasized, however, that the full implementation of CreditMetrics encompasses a broader notion of credit risk, and is likely to produce somewhat larger tail percentiles than our restricted version.

Second, results do not depend very strongly on the distribution of loan sizes within the portfolio, at least within the range of size concentration normally observed in bank portfolios. The discretization of loan sizes in CreditRisk<sup>+</sup> has negligible impact.

Third, both models are highly sensitive to the volatility of default probabilities, or, equivalently, to the average default correlations in the portfolio. When the standard deviation of the default probabilities is doubled, required capital increases by two to three times.

Finally, the models are highly sensitive to the shape of the implied distribution for the systemic risk factors. CreditMetrics, which implies a relatively thin-tailed distribution, reports relatively low tail percentile values for portfolio loss. The tail of CreditRisk<sup>+</sup> depends strongly on the parameter  $\sigma$ , which de-

termines the kurtosis (but not the mean or variance) of the distribution of portfolio loss. Choosing less kurtotic alternatives to the gamma distribution used in CreditRisk<sup>+</sup> sharply reduces its tail percentile values for loss without affecting the mean and variance.

This sensitivity ought to be of primary concern to practitioners. It is difficult enough to measure expected default probabilities and their volatility. Capital decisions, however, depend on extreme tail percentile values of the loss distribution, which in turn depend on higher moments of the distribution of the systemic risk factors. These higher moments cannot be estimated with any precision given available data. Thus, the models are more likely to provide reliable measures for comparing the relative levels of risk in two portfolios than to establish authoritatively absolute levels of capital required for any given portfolio.

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### Appendix A. Properties of the gamma distribution

The gamma distribution is a two parameter distribution commonly used in time-to-failure and other engineering applications. If  $x$  is distributed Gamma( $\alpha, \beta$ ), the probability density function of  $x$  is given by

$$f(x|\alpha, \beta) = \frac{x^{\alpha-1} \exp(-x/\beta)}{\beta^\alpha \Gamma(\alpha)}, \quad (\text{A.1})$$

where  $\Gamma(\alpha)$  is the Gamma function.<sup>29</sup> The mean and variance of  $x$  are given by  $\alpha\beta$  and  $\alpha\beta^2$ , respectively. Therefore, if we impose  $E[x] = 1$  and  $V[x] = \sigma^2$ , then we must have  $\alpha = 1/\sigma^2$  and  $\beta = \sigma^2$ .

<sup>29</sup> Other parameterizations of this distribution are sometimes seen in the literature. This is the parameterization used by CSFP (1997, Eq. 50).

## Appendix B. Estimating the volatility of default probabilities

This appendix demonstrates a simple non-parametric method of estimating the volatility of default probabilities from historical performance data published by Standard & Poor's (Brand and Bahar, 1998, Table 12). Let  $p_\zeta(x_t)$  denote the probability of default of a grade  $\zeta$  obligor, conditional on the realized value  $x_t$  of a systemic risk factor. We need to estimate the unconditional variance  $V[p_\zeta(x)]$ . We assume that the  $x_t$  are serially independent and that obligor defaults are independent conditional on  $x_t$ . Both CreditMetrics and CreditRisk<sup>+</sup> satisfy this framework, though the two models impose different distributional assumptions for  $x$  and functional forms for  $p(x)$ .

For each year in 1981–97 and for each rating grade, S&P reports the number of corporate obligors in its ratings universe on January 1, and the number of obligors who have defaulted by the end of the calendar year. Let  $\hat{d}_{\zeta t}$  be the number of grade  $\zeta$  defaults during year  $t$ , and let  $\hat{n}_{\zeta t}$  denote the number of grade  $\zeta$  obligors at the start of year  $t$ . Let  $\hat{p}_{\zeta t}$  denote the observed default frequency  $\hat{d}_{\zeta t}/\hat{n}_{\zeta t}$ . We assume that the size of the universe  $\hat{n}_{\zeta t}$  is independent of the realization of  $x_t$ .

The general rule for conditional variance is

$$V[y] = E[V[y|z]] + V[E[y|z]]. \quad (\text{B.1})$$

Applied to the problem at hand, we have

$$V[\hat{p}_\zeta] = E[V[\hat{p}_\zeta|x, \hat{n}_\zeta]] + V[E[\hat{p}_\zeta|x, \hat{n}_\zeta]]. \quad (\text{B.2})$$

Obligor defaults are independent conditional on  $x$ , so  $\hat{d}_{\zeta t} \sim \text{Binomial}(\hat{n}_{\zeta t}, p_\zeta(x_t))$ . The expectation of the conditional variance of  $\hat{p}_{\zeta t}$  is therefore given by

$$\begin{aligned} E[V[\hat{p}_\zeta|x, \hat{n}_\zeta]] &= E[V[\hat{d}_\zeta|x, \hat{n}_\zeta]/\hat{n}_\zeta^2] = E[p_\zeta(x)(1 - p_\zeta(x))/\hat{n}_\zeta] \\ &= E[1/\hat{n}_\zeta](E[p_\zeta(x)] - (V[p_\zeta(x)] + E[p_\zeta(x)]^2)) \\ &= E[1/\hat{n}_\zeta](\bar{p}_\zeta(1 - \bar{p}_\zeta) - V[p_\zeta(x)]), \end{aligned} \quad (\text{B.3})$$

where the second equality follows from the formula for the variance of a binomial random variable, the third equality follows from the mutual independence of  $x$  and  $\hat{n}_\zeta$  and the rule  $V[y] = E[y^2] - E[y]^2$ ; and the final equality from  $E[p_\zeta(x)] = \bar{p}_\zeta$ .

Since  $E[\hat{p}_\zeta|x, \hat{n}_\zeta] = p_\zeta(x)$ , the last term in Eq. (B.2) is simply  $V[p_\zeta(x)]$ . Substitute these simplified expressions into Eq. (B.2) and rearrange to obtain

$$V[p_\zeta(x)] = \frac{V[\hat{p}_\zeta] - E[1/\hat{n}_\zeta]\bar{p}_\zeta(1 - \bar{p}_\zeta)}{1 - E[1/\hat{n}_\zeta]}. \quad (\text{B.4})$$

The values of  $\bar{p}_\zeta$  observed in the S&P data differ slightly from the values used for calibration in Section 4.2. It is most convenient, therefore, to normalize the estimated default rate volatilities as ratios of the standard deviation of  $p_\zeta(x)$  to

its expected value,  $\sqrt{V[p_\zeta(x)]}/\bar{p}_\zeta$ . In the first two columns of Table 8, we present for each rating grade the empirical values of  $\bar{p}$  and  $E[1/\hat{n}]$  in the S&P data. The third column presents the observed variance of default rates,  $\hat{V}[\hat{p}]$ , expressed in normalized form. The fourth column gives the implied normalized volatilities for the unobserved true conditional default probabilities.

For the highest grades, AAA and AA, no defaults occurred in the S&P sample, so it is impossible to estimate a volatility for these grades. Among the A obligors, only five defaults were observed in the sample, so the default volatility is undoubtedly measured with considerable imprecision. Therefore, calibration of the normalized volatilities for these grades requires some judgment. Our chosen values for these ratios are given in the final column of Table 8. It is assumed that normalized volatilities are somewhat higher for the top grades, but that the estimated value for grade A is implausibly high. For the lower grades, the empirical estimates are made with greater precision (due to the larger number of defaults in sample), so these values are maintained.

**Appendix C. Proof of Proposition 1**

Let  $y_1$  and  $y_2$  be the CreditMetrics latent variables for two grade  $\zeta$  obligors. Assume that there is only one systemic risk factor and that the two obligors have the same weight  $w_\zeta$  on that risk factor. Thus,

$$y_i = xw_\zeta + \sqrt{1 - w_\zeta^2}\epsilon_i \quad \text{for } i \in \{1, 2\}. \tag{C.1}$$

Conditional on  $x$ , default events for these obligors are independent, so

$$\begin{aligned} \Pr(y_1 < C_\zeta \ \& \ y_2 < C_\zeta | x) &= \Pr(y_1 < C_\zeta | x) \Pr(y_2 < C_\zeta | x) \\ &= \Phi\left(\frac{C_\zeta - xw_\zeta}{\sqrt{1 - w_\zeta^2}}\right)^2 = p_\zeta(x)^2. \end{aligned} \tag{C.2}$$

Table 8  
Empirical default frequency and volatility

	$\bar{p}_\zeta$	$E[1/\hat{n}_\zeta]$	$\sqrt{\hat{V}[\hat{p}_\zeta]}/\bar{p}_\zeta$	$\sqrt{\hat{V}[p_\zeta(x)]}/\bar{p}_\zeta$	$\sqrt{V_\zeta}/\bar{p}_\zeta$
AAA	0	0.0092	–	–	1.4
AA	0	0.0030	–	–	1.4
A	0.0005	0.0017	2.4857	1.5896	1.2
BBB	0.0018	0.0026	1.2477	0.3427	0.4
BB	0.0091	0.0038	1.2820	1.1108	1.1
B	0.0474	0.0041	0.6184	0.5492	0.55
CCC	0.1890	0.0360	0.5519	0.3945	0.4

Therefore,

$$\begin{aligned}\text{Var}[p_{\zeta}(x)] &= E[p_{\zeta}(x)^2] - E[p_{\zeta}(x)]^2 \\ &= E[\text{Pr}(y_1 < C_{\zeta} \ \& \ y_2 < C_{\zeta}|x)] - \bar{p}_{\zeta}^2.\end{aligned}\tag{C.3}$$

Since  $y_1$  and  $y_2$  each have mean zero and variance one, and have correlation  $w_{\zeta}^2$ , the unconditional expectation  $E[\text{Pr}(y_1 < C_{\zeta} \ \& \ y_2 < C_{\zeta}|x)]$  is given by  $\Phi(C_{\zeta}, C_{\zeta}, w_{\zeta}^2)$ . This gives

$$\text{Var}[p_{\zeta}(x)] = \Phi(C_{\zeta}, C_{\zeta}, w_{\zeta}^2) - \bar{p}_{\zeta}^2,\tag{C.4}$$

as required.

#### Appendix D. Distribution of an exponentiated gamma random variable

Assume that  $x$  is distributed such that  $x^{1/r} \sim \text{Gamma}(\alpha, \beta)$  for some given  $r > 0$ . We wish to solve for parameters  $(\alpha, \beta)$  such that  $E[x] = 1$  and  $\text{Var}[x] = \sigma^2$ .

It is straight forward to show that the moments of  $x$  are given by

$$E[x^k] = \frac{\Gamma(\alpha + rk)}{\Gamma(\alpha)} \beta^{rk}.\tag{D.1}$$

Therefore,

$$E[x] = \frac{\Gamma(\alpha + r)}{\Gamma(\alpha)} \beta^r = 1 \quad \Rightarrow \quad \beta = \left( \frac{\Gamma(\alpha)}{\Gamma(\alpha + r)} \right)^{1/r}.\tag{D.2}$$

To solve for  $\alpha$ , use the variance restriction

$$\text{Var}[x] = E[x^2] - E[x]^2 = \sigma^2\tag{D.3}$$

which implies

$$\begin{aligned}\sigma^2 + 1 &= E[x^2] = \frac{\Gamma(\alpha + 2r)}{\Gamma(\alpha)} \beta^{2r} \\ &= \frac{\Gamma(\alpha + 2r)}{\Gamma(\alpha)} \left( \frac{\Gamma(\alpha)}{\Gamma(\alpha + r)} \right)^2 = \frac{\Gamma(\alpha + 2r)\Gamma(\alpha)}{\Gamma(\alpha + r)^2}.\end{aligned}\tag{D.4}$$

For any  $r > 0$ , there exists a unique solution to this equation for  $\alpha$ . To see this, define  $g_r(\alpha)$  as the right-hand side of Eq. (D.4). The gamma function is continuous for non-negative arguments, so  $g_r(\alpha)$  is continuous as well. Note that  $\Gamma(0) = \infty$ , so  $\lim_{\alpha \rightarrow 0} g_r(\alpha) = \infty$  for  $r > 0$ . Using Stirling's formula for  $\Gamma(z)$  as  $z \rightarrow \infty$  (Abramowitz and Stegun, 1968, 6.1.37), it is straightforward to show that  $\lim_{\alpha \rightarrow \infty} g_r(\alpha) = 1$ . Therefore, by the Intermediate Value Theorem, there exists a unique solution to  $g_r(\alpha) = 1 + \sigma^2$ .

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